1. (i) $\int \frac{dx}{x\,(2-3x)} = \int \frac{1}{2}\left(\frac{3}{2-3x} + \frac{1}{x}\right) dx = -\frac{1}{2}\ln|2-3x| + \frac{1}{2}\ln|x| \;+c = \frac{1}{2}\ \ln\left|\frac{x}{2-3x}\right| + c.$

(ii) $\int \frac{dx}{x\,(x^2+1)} = \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx = \ln|x| - \frac{1}{2}\ln|x^2+1| + c = \frac{1}{2}\ln\left|\frac{x^2}{x^2+1}\right| + c.$

(iii) Substitute $x = a\sin\theta$. $\int \left(a^2 - x^2\right)^{1/2} dx = \int a\cos\theta\ a\cos\theta\ d\theta = a^2 \int \frac{1}{2}(\cos 2\theta + 1)\,d\theta$
$= \frac{a^2}{2}\left(\frac{1}{2}\sin 2\theta + \theta\right) + c = \frac{a^2}{2}\left(\sin\theta\cos\theta + \theta\right) + c = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c\,.$

(iv) Substitute $x = u^2$. $\int \frac{x^{1/2}}{1+x}\,dx = \int \frac{2u^2}{1+u^2}\,du = \int 2\left(1 - \frac{1}{1+u^2}\right) du =$
$2u - 2\tan^{-1}u + c = 2x^{1/2} - 2\tan^{-1}x^{1/2} + c\,.$

(v) Substitute $u =$ denominator. (vi) Substitute $x = \tan\theta$. (vii) Substitute $x = e^u$.

(viii) Put $\tan^2 x = \sec^2 x - 1$. (ix) Substitute $u = \tan x$.

(x) $\int \frac{dx}{2+\sin x}$.
Use the half-angle substitution i.e. $t = \tan\frac{x}{2}$, of which details are given in lectures.
Then $\tan x = \frac{2t}{1-t^2}$ so $\sin x = \frac{2t}{1+t^2}$. Also from $t = \tan\frac{x}{2}$, $dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx = \frac{1}{2}(1+t^2)dx\,.$
Hence $\int \frac{dx}{2+\sin x} = \int \frac{2\,dt}{2+2t^2+2t} = \int \frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}}$. Now put $u = t + \frac{1}{2}$, $a = \left(\frac{3}{4}\right)^{1/2}$,
so integral $= \int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c = \left(\frac{4}{3}\right)^{1/2}\tan^{-1}\left\{\left(\frac{4}{3}\right)^{1/2}(t+\frac{1}{2})\right\} + c$
$= \left(\frac{4}{3}\right)^{1/2}\tan^{-1}\left\{\left(\frac{4}{3}\right)^{1/2}(\tan\frac{x}{2} + \frac{1}{2})\right\} + c\,.$

(xi) In order to remove x^2 from the integrand, use integration by parts
($\int u\,dv = uv - \int v\,du$) twice, firstly with $u = x^2$ and $v = \sin x$.

(xii) Put $I = \int e^x \cos x\,dx$. Use integration by parts twice. Firstly, with $u = e^x$ and $v = \sin x$, $\int u\,dv = uv - \int v\,du$ gives $I = e^x\sin x - \int e^x \sin x\,dx$.
Next, applying integration by parts to the last integral with $u = e^x$ and $v = -\cos x$,
$I = e^x\sin x - \left\{e^x(-\cos x) - \int e^x(-\cos x)\,dx\right\} = e^x\sin x + e^x\cos x - I\,.$
Hence $2I = e^x(\sin x + \cos x) + c$, giving the required result.
Alternativly, use a result from Question 8 on Sheet 10.

(xiii) In order to remove x from the integrand, use integration by parts.
So take $u = x$ and $dv = \frac{\cos x}{\sin^2 x}\,dx$, so that $v = -\frac{1}{\sin x}$.

(xiv) Take $u = \ln x$ and $dv = x^k\,dx$, so that $v = \frac{x^{k+1}}{k+1}$.

(xv) Complete the square. Then substitute $u = x + 1$.

$$\int_{-1}^{1} \frac{dx}{x^2+2x+5} = \int_{-1}^{1} \frac{dx}{(x+1)^2+4}\,dx = \int_{0}^{2} \frac{du}{u^2+4}\,dx = \left[\frac{1}{2}\tan^{-1}\frac{u}{2}\right]_0^2 = \pi/8\,.$$

(xvi) Note $1 + \cos x = 2\cos^2(x/2)$ and then write $\sec^4(x/2) = (1 + \tan^2(x/2))\sec^2(x/2)$.

(xvii) Complete the square and substitute $x + 2 = 3\tan\theta$.

P.T.O.

2. $I_{n+1} \;=\; \displaystyle\int_0^\infty x^{n+1}\,\mathrm{e}^{-x^2}\,dx\,.$ Use integration by parts.

In $\int u\,dv = uv \;-\; \int v\,du\,,$ put $u = x^n$ and $dv = x\,\mathrm{e}^{-x^2}\,dx$ so that $v = -\frac{1}{2}\mathrm{e}^{-x^2}$, and then

$$I_{n+1} \;=\; \int u\,dv \;=\; \left[x^n\left(-\frac{1}{2}\mathrm{e}^{-x^2}\right)\right]_0^\infty \;-\; \int_0^\infty nx^{n-1}\left(-\frac{1}{2}\mathrm{e}^{-x^2}\right)dx \;=\; 0 \;+\; \frac{n}{2}I_{n-1}.$$

Hence $I_{n+1} \;=\; \frac{n}{2}I_{n-1}\,.$

Putting $n = 4$, and then 2, in the last equation gives

$I_5 \;=\; \frac{4}{2}I_3 \;=\; \frac{4}{2}\frac{2}{2}I_1$, where $I_1 = \left[-\frac{1}{2}\mathrm{e}^{-x^2}\right]_0^\infty \;=\; \frac{1}{2}$, and so $I_1 = 1$.

3. $u_n + iv_n \;=\; \int x^n\,(\cos x + i\sin x)\,dx \;=\; \int x^n\,\mathrm{e}^{ix}\,dx \;=\; x^n\,\left(-i\mathrm{e}^{ix}\right) \;-\; \int nx^{n-1}\,\left(-i\mathrm{e}^{ix}\right)\,dx$

Hence $u_n + iv_n \;=\; -ix^n\,\mathrm{e}^{ix} \;+\; in(u_{n-1} + iv_{n-1})\,.$ Taking real and imaginary parts,

$$u_n \;=\; x^n\sin x \;-\; nv_{n-1}\,, \qquad\qquad v_n \;=\; -x^n\cos x \;+\; nu_{n-1}$$

Hence $v_4 \;=\; -x^4\cos x \;+\; 4u_3\;,\quad u_3 \;=\; x^3\sin x \;-\; 3v_2,\quad v_2 \;=\; -x^2\cos x \;+\; 2u_1,$

$u_1 \;=\; x\sin x \;-\; v_0,\quad v_0 \;=\; -\cos x\,,$ giving

$v_4 \;=\; -x^4\cos x \;+\; 4x^3\sin x \;+\; 12x^2\cos x \;-\; 24x\sin x \;-\; 24\cos x + c\,.$

4. $I_n \;=\; \int_0^{\pi/4}\tan^n x\,dx\,.$ Write $\tan^n x \;=\; \tan^{n-2}x\,\tan^2 x \;=\; \tan^{n-2}x\,(\sec^2 x - 1)\,,$ and note

$\tan^{n-2}x\,\sec^2 x \;=\; \frac{d}{dx}\left(\frac{1}{n-1}\tan^{n-1}x\right)$, so that $I_n \;=\; \left[\frac{1}{n-1}\tan^{n-1}x\right]_0^{\pi/4} \;-\; I_{n-2}\,.$

Hence $I_n \;=\; \frac{1}{n-1} \;-\; I_{n-2}\,.$

With $n = 5$, $I_5 \;=\; \frac{1}{4} \;-\; I_3 \;=\; \frac{1}{4} \;-\; (\frac{1}{2} \;-\; I_1\,)$, where $I_1 \;=\; [\,-\ln|\cos x|\,]_0^{\pi/4} \;=\; \frac{1}{2}\ln 2\,.$

Hence $I_5 \;=\; \frac{1}{2}\ln 2 - \frac{1}{4}\,.$