1. (i) Consider $\lim_{\lambda\to\infty}\int_3^\lambda x^{-3/2}dx = \lim_{\lambda\to\infty}\left[-2\,x^{-1/2}\right]_{x=3}^{\lambda} = \lim_{\lambda\to\infty}\left[2(3)^{-\frac{1}{2}} - 2\,\lambda^{-\frac{1}{2}}\right] = 2\,(3)^{-\frac{1}{2}}$.

(ii) Substsituting $t = x-1$, integral $= \int_0^1 t^{-2/3}\,dt$ so consider

$\lim_{\varepsilon\to 0}\int_\varepsilon^1 t^{-2/3}\,dt = \lim_{\varepsilon\to 0}\left[\,3\,t^{1/3}\,\right]_\varepsilon^0 = \lim_{\varepsilon\to 0}\left(\,3 - 3\,\varepsilon^{1/3}\,\right) = 3\,.$

(iii) Consider $\lim_{\varepsilon\to 0}\int_\varepsilon^1 \ln x\,dx = \lim_{\varepsilon\to 0}[x\ln x - x]_\varepsilon^1 = \lim_{\varepsilon\to 0}(\ln 1 - 1 - \varepsilon\ln\varepsilon + \varepsilon) = -1\,.$

(iv) Consider $\lim_{\lambda\to\infty}\int_1^\lambda \ln x\,dx = \lim_{\lambda\to\infty}[x\ln x - x]_1^\lambda = \lim_{\lambda\to\infty}\{\lambda(\ln\lambda - 1) - \ln 1 - 1\}$,
which does not exist *finitely*, so the integral diverges.

(v) Consider $\lim_{\lambda\to\pi/2}\int_0^\lambda \tan x\,dx = \lim_{\lambda\to\pi/2}[-\ln|\cos x|]_0^\lambda = \lim_{\lambda\to\pi/2}\{-\ln|\cos\lambda| + \ln 1\}$,
which does not exist *finitely*, so the integral diverges.

2. (i) 0, since $\sinh x^3$ is an odd function.
(ii) $\frac{2}{5}\,(1-e^{-15})$. The integrand is even, so integral $= 2\int_0^3 e^{-5x}\,dx$.

3. Length $= \int_0^1\left\{(dx)^2 + (dy)^2\right\}^{1/2} = \int_0^1\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{1/2} dx = \int_0^1\left\{1+\sinh^2 x\right\}^{1/2} dx$
$= \int_0^1 \cosh x\,dx = \sinh 1\,.$

4. $\frac{dy}{dx} = \frac{x}{2} - \frac{1}{2x}$ so length $= \int_1^3\left\{1 + \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}\right\}^{1/2} dx = \int_1^3 \frac{1}{2x}\left\{x^4 + 2x^2 + 1\right\}^{1/2} dx$
$= \int_1^3 \frac{1}{2x}\left\{x^2+1\right\} dx = \left[\frac{x^2}{4} + \frac{1}{2}\ln x\right]_1^3 = 2 + \frac{1}{2}\ln 3\,.$

5. Length of the cable $= \int_0^{2L}\left\{1 + (y')^2\right\}^{1/2} dx\,.$ $y' = \frac{2H}{L^2}\,x - \frac{2H}{L} = \frac{2H}{L}\left(\frac{x}{L} - 1\right)$.

Hence length $= \int_0^{2L}\left\{1 + 4\left(\frac{H}{L}\right)^2\left(\frac{x}{L}-1\right)^2\right\}^{1/2} dx\,.$

Substitute $\sinh u = 2\left(\frac{H}{L}\right)\left(\frac{x}{L}-1\right)$ so $\cosh u\,du = \left(\frac{2H}{L^2}\right) dx$, giving length

$$= \frac{L^2}{2H}\int_{-u_0}^{u_0}\cosh^2 u\,du\,, \qquad \text{where} \quad \sinh u_0 = \frac{2H}{L}\,,$$

$$= \frac{L^2}{4H}\int_{-u_0}^{u_0}(1+\cosh 2u)\,du = \frac{L^2}{2H}\left[u + \frac{1}{2}\sinh 2u\right]_{-u_0}^{u_0}$$

$$= \frac{L^2}{2H}\left\{\sinh^{-1}\frac{2H}{L} + \frac{2H}{L}\left(1 + \frac{4H^2}{L^2}\right)^{1/2}\right\}$$

$$= \frac{L^2}{2H}\sinh^{-1}\frac{2H}{L} + (L^2 + 4H^2)^{1/2}\,.$$

P.T.O.

Putting $\varepsilon = 2H/L$, length

$$= \frac{L}{\varepsilon}\,\sinh^{-1}\varepsilon \;+\; L\,(1+\varepsilon^2)^{1/2} \;=\; \frac{L}{\varepsilon}\left(\varepsilon - \frac{\varepsilon^3}{3!} + \ldots\right) \;+\; L\left(1 + \frac{\varepsilon^2}{2} + \ldots\right)$$

$$= 2L \;+\; \frac{L\varepsilon^2}{3} \;+\; \ldots \;\approx\; 2L\left\{1 \;+\; \frac{2}{3}\left(\frac{H}{L}\right)^2\right\}.$$

6. Length $= \displaystyle\int ds \;=\; \int \left\{(dx)^2 + (dy)^2\right\}^{1/2} \;=\; \int_0^\pi \left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\}^{1/2} d\theta$

$$= \int_0^\pi \left\{(1+\cos\theta)^2 + (-\sin\theta)^2\right\}^{1/2} d\theta \;=\; \int_0^\pi 2^{1/2}\,(1+\cos\theta)^{1/2}\;d\theta$$

$$= \int_0^\pi 2\,\cos(\theta/2)\,d\theta \;=\; 4\,[\,\sin(\theta/2)\,]_0^\pi \;=\; 4\,.$$