1. (i) $y = (c+x)\sin x$, so $y' = (c+x)\cos x + \sin x$ giving $y' = y\cot x + \sin x$.
(ii) $x = c\exp\left(-\frac{x}{y}\right)$, so $c = x\exp\left(\frac{x}{y}\right)$, $0 = \exp\left(\frac{x}{y}\right) + x\left(\frac{1}{y} - x\frac{y'}{y^2}\right)\exp\left(\frac{x}{y}\right)$.
Hence $0 = 1 + \frac{x}{y} - \left(\frac{x}{y}\right)^2 y'$ so that $y' = \frac{y}{x} + \left(\frac{y}{x}\right)^2$.

2. (i) Separable : $\dfrac{dy}{y+1} = \dfrac{dx}{x+1}$, so $\ln|y+1| = \ln|x+1| + c$.
But $y(0) = 1$, giving $\ln 2 = \ln 1 + c$. Hence $c = \ln 2$, $y+1 = 2(x+1)$, $y = 2x+1$.
(ii) Separable : $\frac{dy}{y} = -\frac{4x\,dx}{1+x^2}$, so $\ln|y| = -2\ln|1+x^2| + c$.
But $y(0) = \frac{1}{2}$, giving $\ln\frac{1}{2} = -2\ln 1 + c$. Hence $c = -\ln 2$, $y = 1/(2(1+x^2)^2)$.
(iii) Separable : $\frac{dy}{1-y} = \frac{x\,dx}{1-x^2}$, so $-\ln|1-y| = -\frac{1}{2}\ln|1-x^2| + c$.
But $y(0) = 2$, so $\ln|1| = -\frac{1}{2}\ln|1| + c$. Hence $c = 0$, $(1-y)^2 = (1-x^2)$, $y = 1 \pm (1-x^2)^{\frac{1}{2}}$.
(iv) Not separable, but homogeneous : $\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 - \frac{y}{x}$. So put $v(x) = \frac{y}{x}$ i.e. $y(x) = x\,v(x)$.
Then $\frac{dy}{dx} = x\frac{dv}{dx} + v$ and the equation becomes
$x\dfrac{dv}{dx} + v = v^2 - v$ so $x\dfrac{dv}{dx} = v(v-2)$. Now separable, and using partial fractions,
$\int \frac{dx}{x} = \frac{1}{2}\int\left(\frac{1}{v-2} - \frac{1}{v}\right) dv$. Hence $2\ln x + 2c = \ln|v-2| - \ln|v| = \ln\left|1 - \frac{2}{v}\right|$.
So $1 - \frac{2x}{y} = k\,x^2$ where $k = e^{2c}$. $y = 2x\left(1 - k\,x^2\right)^{-1}$.
(v) Similar to (iv), now with $\frac{dy}{dx} = -\frac{3}{2}\frac{x}{y} - \frac{1}{2}\frac{y}{x}$.
(vi) Again homogeneous $\frac{dy}{dx} = \frac{2x^3y - y^4}{x^4 - 2xy^3} = \frac{2v - v^4}{1-2v^3}$, where $v(x) = \frac{y}{x}$, then separable.
(vii) Not homogeneous or separable, but linear, so use integrating factor.
I.F. $= \exp\int 2\tan x\,dx = \exp(-2\ln\cos x) = (\cos x)^{-2}$.
Multiplying by the I.F.,
$\frac{d}{dx}\left\{(\cos x)^{-2}\,y\right\} = \sin x\,(\cos x)^{-2}$, so that $(\cos x)^{-2}\,y = (\cos x)^{-1} + c$,
$y = (1 + c\cos x)\cos x$. But $y(\pi) = -3$, so $-3 = c - 1$, $c = -2$. $y = (1 - 2\cos x)\cos x$.
(viii) Again not homogeneous or separable, but linear, so use integrating factor.
I.F. $= \exp\int 2x\,dx = \exp(x^2)$. Multiplying by this I.F., proceed to solve $\frac{d}{dx}\left\{e^{x^2}\,y\right\} = 2$.
(ix) Again linear, but NOTE to rearrange as $\frac{dy}{dx} + \frac{2}{x}y = x - 1 + x^{-1}$, before using I.F. method. I.F. $= \exp\left(\int \frac{2}{x}\right) dx = \exp(2\ln x) = x^2$, giving

$$\frac{d}{dx}\left(x^2\,y\right) = x^3 - x^2 + x, \qquad x^2\,y = \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + c.$$

But $y(1) = \frac{1}{3}$, so that $\frac{1}{3} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c$, so $c = -\frac{1}{12}$. Hence $y = \frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{2} - \frac{1}{12}x^{-2}$.

3. With $x = X + A$ and $y = Y + B$, the ODE becomes $\dfrac{dY}{dX} = \dfrac{2X + 2Y + 2A + 2B - 2}{3X + Y + 3A + B - 5}$.
Therefore choose A and B so that both $2A + 2B - 2 = 0$ and $3A + B - 5 = 0$, i.e. $A = 2$ and $B = -1$. This makes the equation for $Y(X)$ homogeneous in X and Y, so put $Y(X) = X\,V(X)$ to find that

$$X\frac{dV}{dX} + V = \frac{2 + 2V}{3 + V}, \text{ giving } X\frac{dV}{dX} = \frac{2 + 2V - 3V - V^2}{3 + V} = \frac{(2+V)(1-V)}{3+V},.$$

Using partial fractions, $\displaystyle\int \frac{dX}{X} = \frac{1}{3}\int \left(\frac{4}{1-V} + \frac{1}{2+V}\right) dV$.

Hence $3\ln|X| = -4\ln|1-V| + \ln|2+V| + c$, so $X^3\,(1-V)^4\,(2+V)^{-1} = k$,
i.e. $(X-Y)^4 = k\,(2X+Y)$ where $X = x - 2$ and $Y = y + 1$,
so that $(x - y - 1)^4 = k\,(2x + y - 3)$.

4. (i) The ODE is exact if $\dfrac{\partial}{\partial y}(9x^2 + y - 1) = \dfrac{\partial}{\partial x}(x - 4y)$ i.e. if $1 = 1$.

Hence there exists a function $\phi(x,y)$ such that $\dfrac{\partial \phi}{\partial x} = 9x^2 + y - 1$ and $\dfrac{\partial \phi}{\partial y} = x - 4y$,
so that $\phi(x,y) = \text{constant}$ is the general solution of the ODE.
Using partial integation of $\dfrac{\partial \phi}{\partial x} = 9x^2 + y - 1$, $\quad \phi(x,y) = 3x^3 + xy - x + g(y)$.
Then $\dfrac{\partial \phi}{\partial y} = \dfrac{\partial}{\partial y}(3x^3 + xy - x + g(y)) = x - 4y$ giving $g'(y) = -4y$, $g(y) = -2y^2$,
and hence solution $\qquad \phi(x,y) = 3x^3 + xy - x - 2y^2 = \text{constant}.$

(ii) This ODE is exact if $\dfrac{\partial}{\partial y}(xy^2 + y) = \dfrac{\partial}{\partial x}(x^2y + x)$ i.e. if $2xy + 1 = 2xy + 1$. Then use the same method as in (i).

5. The ODE becomes $(5x^{2+n} + 12x^{1+n}y - 3x^n y^2)\,dx + (3x^{2+n} - 2x^{1+n}y)\,dy = 0$.
The ODE is exact if $\dfrac{\partial}{\partial y}(5x^{2+n} + 12x^{1+n}y - 3x^n y^2) = \dfrac{\partial}{\partial x}(3x^{2+n} - 2x^{1+n}y)$
i.e. if $12x^{1+n} - 6x^n y = 3(2+n)x^{1+n} - 2(1+n)x^n y$. Hence, on comparing coefficients, one needs

$$12 = 3(2+n) \text{ AND } -6 = -2(1+n), \quad \text{so that } n = 2.$$

Hence $\phi(x,y) = \text{constant}$ is the general solution of the ODE, where
$\dfrac{\partial \phi}{\partial x} = 5x^4 + 12x^3y - 3x^2y^2$ and $\dfrac{\partial \phi}{\partial y} = 3x^4 - 2x^3y$.
Using partial integation of $\dfrac{\partial \phi}{\partial x} = 5x^4 + 12x^3y - 3x^2y^2$, $\quad \phi(x,y) = x^5 + 3x^4y - x^3y^2 + g(y)$.
Then $\dfrac{\partial \phi}{\partial y} = 3x^4 - 2x^3y$ giving $g'(y) = 0$, $g(y) = 0$,
and hence solution $\qquad \phi(x,y) = x^5 + 3x^4y - x^3y^2 = \text{constant}.$

6. Multiply the ODE by $f(z)$ where $z = xy^2$. The equation is exact if

$$\frac{\partial}{\partial y} f(z)\,(y^4 - 2y^2) \;=\; \frac{\partial}{\partial x} f(z)\,(3xy^3 - 4xy + y)$$

i.e., since $\dfrac{\partial}{\partial y} f(z) \;=\; x2yf'(z)$ and $\dfrac{\partial}{\partial x} f(z) \;=\; y^2 f'(z)$,

$$(y^4 - 2y^2)\,2xyf' \;+\; (4y^3 - 4y)\,f \;=\; (3xy^3 - 4xy + y)\,y^2 f' \;+\; (3y^3 - 4y + 0)\,f$$

giving $f'\,(xy^2 + 1) \;=\; f$ so that $f(z)$, with $z = xy^2$, must satisfy $f'(z)\,(z + 1) \;=\; f(z)$.
Hence take $f(z) \;=\; (1 + z)$.

Hence $\phi(x, y) = \text{constant}$ is the general solution of the ODE, where

$\dfrac{\partial \phi}{\partial x} = (1 + xy^2)\,(y^4 - 2y^2)$ i.e. $\dfrac{\partial \phi}{\partial x} = y^4 - 2y^2 + xy^6 - 2xy^4$ and

$\dfrac{\partial \phi}{\partial y} = (1 + xy^2)\,(3xy^3 - 4xy + y)$, i.e. $\dfrac{\partial \phi}{\partial y} = 3xy^3 - 4xy + y + 3x^2y^5 - 4x^2y^3 + xy^3$.

Using partial integation of $\dfrac{\partial \phi}{\partial x}$, $\quad \phi(x, y) = \; xy^4 - 2xy^2 + \tfrac{1}{2}x^2y^6 - x^2y^4 + g(y)$.

Then $\dfrac{\partial}{\partial y}(xy^4 - 2xy^2 + \dfrac{1}{2}x^2y^6 - x^2y^4 + g(y)) \;=\; 3xy^3 - 4xy + y + 3x^2y^5 - 4x^2y^3 + xy^3$

giving $g'(y) = y$, and take $g(y) = \tfrac{1}{2}y^2$,

and hence solution $\qquad \phi(x, y) = xy^4 - 2xy^2 + \tfrac{1}{2}x^2y^6 - x^2y^4 + \tfrac{1}{2}y^2 = \text{constant}.$

7. When the fluid is at depth y, the volume V of liquid is $V = \dfrac{1}{3}\pi r^2 y$, where $r = \dfrac{y}{H}R$.

Hence $V = \dfrac{\pi R^2}{3H^2}\,y^3$. $\quad$ The rate of change of volume is

$$\frac{dV}{dt} = -\,(\text{velocity of escaping fluid}) \times (\text{area of hole}) \;=\; -\,ky^{1/2}\,\pi a^2 .$$

Hence $\qquad \dfrac{dV}{dt} \;=\; \dfrac{\pi R^2}{3H^2}\,3y^2\dfrac{dy}{dt} \;=\; -\,ky^{1/2}\,\pi a^2$, giving $y^{3/2}\,\dfrac{dy}{dt} \;=\; -\,\dfrac{kH^2a^2}{R^2}$.

Solving, $\dfrac{2}{5}\,y^{5/2} \;=\; -\,\dfrac{kH^2a^2}{R^2}\,t \;+\; c.$

For $t = 0,\; y = H$, so that $c = \dfrac{2}{5}\,H^{5/2}$. $\quad$ Then, when $y = 0,\; t \;=\; c\,\dfrac{R^2}{kH^2a^2} \;=\; \dfrac{2}{5}\,\dfrac{R^2H^{1/2}}{ka^2}$.