BioFluids Lecture 16: Steady flows for various Dean numbers

We have just derived the steady Dean equations for fully-developed flow down a slightly curved pipe, locally in the y-direction:

$$\left.\begin{aligned} K(\mathbf{u}\cdot\nabla v) &\equiv K(\psi_z v_x - \psi_x v_z) = 1 + \nabla^2 v \\ K(\mathbf{u}\cdot\nabla\Omega) &\equiv K(\psi_z \Omega_x - \psi_x \Omega_z) = \nabla^2\Omega - 2Kvv_z \end{aligned}\right\} \tag{16.1}$$

where $\Omega = -\nabla^2\psi$. For a pipe of circular cross-section in the $(x,\, z)$ plane it is natural to use polar coordinates $(R,\, \theta)$ with $x = R\sin\theta$ and $z = R\cos\theta$. In terms of a streamfunction, ψ, the velocity $\mathbf{u} = (0,\, v(R,\, \theta),\, 0) + \nabla \wedge (0,\, \psi(R,\, \theta),\, 0)$ giving

$$\left.\begin{aligned} &\frac{K}{R}(\psi_\theta v_R - \psi_R v_\theta) = 1 + \nabla^2 v \\ &\frac{K}{R}(\psi_\theta \Omega_R - \psi_R \Omega_\theta) = \nabla^2\Omega - 2Kv\left(v_R\sin\theta + \frac{v_\theta\cos\theta}{R}\right) \end{aligned}\right\} \tag{16.2}$$

where

$$-\Omega = \nabla^2\psi = \psi_{RR} + \frac{\psi_R}{R} + \frac{\psi_{\theta\theta}}{R^2}$$

is the downpipe vorticity. The no-slip boundary conditions on $R = 1$ require

$$\psi = 0, \qquad \psi_R = 0 \qquad v = 0 \qquad \text{on the pipe boundary.} \tag{16.3}$$

The Dean number, K, is a Reynolds number modified by the pipe curvature. We investigate solutions to these equations for low, high and intermediate values of K.

Low Dean number: When $K = 0$, the solution is just the Poiseuille flow,

$$v = v_0(R) \equiv \tfrac{1}{4}(1 - R^2). \tag{16.4}$$

Substituting into (16.3), we see this drives at $O(K)$ a cross-flow given by

$$-\nabla^2(\nabla^2\psi) \equiv \nabla^2\Omega = 2K\sin\theta v_0(R)v_0'(R)$$

with solution for a known constant A

$$\psi = \psi_1(R,\theta) \equiv AR(1 - R^2)^2(4 - R^2)\sin\theta. \tag{16.5}$$

This drives an $O(K^2)$ correction to the downpipe flow $\nabla^2 v_2 = v_0'(R)\psi_{1\theta}/R$

$$v = v_0(R) + K^2 v_2(R,\theta) \qquad \text{where} \quad v_2 \equiv B\cos\theta R(1 - R^2)(19 - 21R^2 + 9R^4 - R^6)$$

This process can be repeated to derive the series expansion

$$v = \sum_{n=0}^{\infty} K^{2n} v_n(R,\, \theta), \qquad \psi = \sum_{n=0}^{\infty} K^{2n+1}\psi_n(R,\, \theta) \tag{16.6}$$

In principle, all these functions could be found analytically. They take the form

$$v_n = \sum_{j=0}^{n} \cos(j\theta) P_{nj}(R) \qquad \text{where} \quad P_{nj}(R) = \sum_i C_{nij} R^i \tag{16.7}$$

is a polynomial in R, and a similar expression for ψ_n. The first two terms in these expansions illustrate the leading order effects of curvature. The shape of ψ_1 is famous: it gives rise to two counter-rotating "Dean vortices." The first correction $O(K^2)$ to the downpipe flow moves the maximum of v towards the outside of the bend. See figures.

Van Dyke (1978) calculated 24 terms in the series (16.6) and used series extension techniques to increase the domain of convergence. Recently Tettamanti (2009) has calculated 60 terms, extending and correcting Van Dyke's conclusions.

Moderate Dean numbers: numerical solutions

For higher Dean numbers, the equations should be solved numerically. See Collins & Dennis (1975), Dennis & Ng (1982), Siggers & Waters (2005). A useful review of Dean flows can be found in Berger et al (1983). Broadly speaking, ψ_1 and v_1 predict well the behaviour of the flow as K increases. The position of maximum v migrates towards the outside, while the contours of v become less z-dependent, except near the wall. The closed ψ regions lose their left/right symmetry, and their contours become closer near the wall, despite the double zero enforced by the no-slip condition. As K increases further it becomes clearer that a boundary layer structure emerges. It has some similarity with the 'Ekman layers' associated with rapidly rotating flows. The wall shear stress is minimum on the inside of the bend.

The high Dean number limit:

As $K \to \infty$ a solution with an inviscid core and a thin boundary layer structure is discernible. We can estimate the asymptotic scalings as follows. Let v and ψ have scales V_0 and Ψ_0. Then if the boundary layer has thickness δ the secondary velocity in the layer is of order Ψ_0/δ whereas in the core it is of order $\Psi_0/1$. In the layer the vorticity $\Omega \sim \Psi_0/\delta^4$.

In the inviscid core, the leading order terms in (16.2) are

$$K\mathbf{u} \cdot \nabla v = 1, \qquad 0 = v v_z \tag{16.8}$$

giving the scalings and structure

$$K\Psi_0 V_0 = 1 \qquad v = v_c(r), \qquad \psi = z/K v_c'(r) \tag{16.9}$$

Note that the z dependence fits with the pattern of the numerical solutions. In the layers, since $\mathbf{u}$ is larger we have to balance $K\mathbf{u} \cdot \nabla$ with ∇^2 and the driving term $v v_z$, giving the scales

$$K\frac{\Psi_0}{\delta} = \frac{1}{\delta^2} = \frac{K V_0^2}{\delta} \tag{16.10}$$

Combining (16.10) and (16.9) we find

$$\delta = K^{-1/3}, \qquad V_0 = K^{-1/3} \qquad \Psi_0 = K^{-2/3} \tag{16.11}$$

Note this doesn't mean that the flow is asymptotically small. When we nondimensionalised we scaled v with Ga^2/μ; we are predicting now that it should scale with $\mu^{-2/3}$ as $\mu \to 0$. If we introduce a scaled normal coordinate $n = (1-R)/\delta$ equations (16.2) become in the layer

$$\psi_n v_\theta - \psi_\theta v_n = v_{nn}, \tag{16.12}$$

$$\psi_n \psi_{nn\theta} - \psi_\theta \psi_{nnn} = \psi_{nnnn} + v v_n \sin\theta, \tag{16.13}$$

which integrates to

$$\psi_n \psi_{n\theta} - \psi_\theta \psi_{nn} = \psi_{nnn} + \tfrac{1}{2}\sin\theta\left[v^2 - v_c^2\right] \tag{16.14}$$

The boundary conditions are

$$\psi = \psi_n = v = 0 \qquad \text{on } n = 0, \qquad \psi_n \to 0, \quad v \to v_c \qquad \text{as} \quad n \to \infty \tag{16.15}$$

A slightly unusual boundary layer problem emerges, with the core flow determined by a regularity condition on the layer.

The boundary layer starts at $\theta = 0$, at the outer bend, then grows inwards along the top and bottom of the pipe towards the inner bend. It is found that only some matching conditions with v_c are compatible with the boundary layer remaining attached. Physically, one can think of an initial value problem, with zero inital core vorticity ($v \sim 1/r$) as the flow develops, the boundary layers separate off advecting vorticity into the core. Only when a suitable core vorticity distribution is set up does equilibration occur. Numerical solution shows that the equilibrium $v_c(r)$ is more or less linear in r, increasing outwards. Numerical results also agree well with the boundary layer scalings.

However, the precise details of the asymptotics are still unknown. A suitable $v_c(r)$ can regularise the boundary layers until very close to the inner wall Dennis & Riley (1991), but no consistent picture at the inner wall has been found.

Nonuniqueness: the four-gyre solution.

Numerically, it is found that for moderately high K, more than one solution of (5.1) is possible. A four-vortex flow was found by Dennis & Ng, as in the figure. This solution is unstable to perturbations which break top-down symmetry Daskopoulos & Lenhoff (1989). The high Dean number solution can also undergo Hopf bifurcation to a time-dependent oscillation.

Effects of curvature

The Dean equations are mathematically appealing because the Reynolds number and curvature combine to form a single determining parameter. Curvature terms can be retained in the model, even if they are regarded as small.

A double expansion in the Dean number and curvature was performed by (Topakoglu 1967) and by Siggers & Waters (2005). The latter also solved the flow numerically, and demonstrated that even small curvature can have noticeable influence on the flow characteristics.