Civ.Eng. 2 Mathematics: Grad, Div and Curl

This sheet can be found on the Web: http://www.ma.ic.ac.uk/∼ajm8/Civ2

A function $\underline{\mathbf{F}}$ which associates a vector value $\underline{\mathbf{F}}(\underline{\mathbf{r}}) \equiv \underline{\mathbf{F}}(x,y,z) \equiv F_1(\underline{\mathbf{r}})\underline{\mathbf{i}} + F_2(\underline{\mathbf{r}})\underline{\mathbf{j}} + F_3(\underline{\mathbf{r}})\underline{\mathbf{k}}$ with every position vector $\underline{\mathbf{r}} \equiv x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\mathbf{k}}$ is said to be a **VECTOR FIELD**, e.g. velocity of a fluid.

Recall if $f(\underline{\mathbf{r}})$ is a scalar field, then

$$\text{the gradient of } f \equiv \text{grad}\, f \equiv \nabla f \equiv \frac{\partial f}{\partial x}\underline{\mathbf{i}} + \frac{\partial f}{\partial y}\underline{\mathbf{j}} + \frac{\partial f}{\partial z}\underline{\mathbf{k}}. \tag{1}$$

$\nabla \equiv \underline{\mathbf{i}}\frac{\partial}{\partial x} + \underline{\mathbf{j}}\frac{\partial}{\partial y} + \underline{\mathbf{k}}\frac{\partial}{\partial z}$ is a **VECTOR OPERATOR**, $\quad \nabla$: scalar field $\rightarrow$ vector field.

Now introduce

$$\text{the divergence of } \quad \underline{\mathbf{F}} \equiv \text{div}\, \underline{\mathbf{F}} \equiv \nabla \cdot \underline{\mathbf{F}} \equiv \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \tag{2}$$

Note that

$$\nabla \cdot \underline{\mathbf{F}} \not\equiv \underline{\mathbf{F}} \cdot \nabla \equiv F_1\frac{\partial}{\partial x} + F_2\frac{\partial}{\partial y} + F_3\frac{\partial}{\partial z},$$

which is the scalar operator we met before when finding directional derivatives; i.e.

$$\underline{\mathbf{F}} \cdot (\nabla f) \equiv F_1\frac{\partial f}{\partial x} + F_2\frac{\partial f}{\partial y} + F_3\frac{\partial f}{\partial z} \equiv (\underline{\mathbf{F}}\,.\,\nabla) f\,.$$

Similarly, we introduce

$$\begin{aligned} \text{curl}\, \underline{\mathbf{F}} \quad &\equiv \quad \nabla \times \underline{\mathbf{F}} \equiv \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &\equiv \quad \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\underline{\mathbf{i}} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\underline{\mathbf{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\underline{\mathbf{k}}. \end{aligned} \tag{3}$$

The equivalent notation curl $\underline{\mathbf{F}} \equiv \nabla \wedge \underline{\mathbf{F}}$ is sometimes used.
Note that $\nabla \times \underline{\mathbf{F}} \not\equiv \underline{\mathbf{F}} \times \nabla$, which is a vector operator; i.e.

$$\begin{aligned} \underline{\mathbf{F}} \times (\nabla f) \quad &\equiv \quad \left(F_2\frac{\partial f}{\partial z} - F_3\frac{\partial f}{\partial y}\right)\underline{\mathbf{i}} - \left(F_1\frac{\partial f}{\partial z} - F_3\frac{\partial f}{\partial x}\right)\underline{\mathbf{j}} + \left(F_1\frac{\partial f}{\partial y} - F_2\frac{\partial f}{\partial x}\right)\underline{\mathbf{k}} \\ &\equiv \quad (\underline{\mathbf{F}} \times \nabla) f\,. \end{aligned}$$

Therefore we have that

$$\begin{aligned} \text{grad} &: \text{scalar field} \quad \to \quad \text{vector field} \\ \text{div} &: \text{vector field} \quad \to \quad \text{scalar field} \\ \text{curl} &: \text{vector field} \quad \to \quad \text{vector field} \end{aligned}$$

RULES: For arbitrary scalar fields $f(\underline{\mathbf{r}})$, $g(\underline{\mathbf{r}})$ and vector fields $\underline{\mathbf{F}}(\underline{\mathbf{r}})$, $\underline{\mathbf{G}}(\underline{\mathbf{r}})$; then we have that

(1) $\implies$ $\nabla(f+g) \equiv (\nabla f) + (\nabla g)\,.$

(2) $\implies$ $\nabla \cdot (\underline{\mathbf{F}} + \underline{\mathbf{G}}) \equiv (\nabla \cdot \underline{\mathbf{F}}) + (\nabla \cdot \underline{\mathbf{G}})\,.$

(3) $\implies$ $\nabla \times (\underline{\mathbf{F}} + \underline{\mathbf{G}}) \equiv (\nabla \times \underline{\mathbf{F}}) + (\nabla \times \underline{\mathbf{G}})\,.$

(2) and (1) $\implies$

$$\begin{aligned} \nabla \cdot (f\,\underline{\mathbf{F}}) &\equiv \nabla \cdot \left[(f\,F_1)\underline{\mathbf{i}} + (f\,F_2)\underline{\mathbf{j}} + (f\,F_3)\underline{\mathbf{k}} \right] \equiv \frac{\partial (f\,F_1)}{\partial x} + \frac{\partial (f\,F_2)}{\partial y} + \frac{\partial (f\,F_3)}{\partial z} \\ &\equiv \left[F_1 \frac{\partial f}{\partial x} + f\,\frac{\partial F_1}{\partial x} \right] + \left[F_2 \frac{\partial f}{\partial y} + f\,\frac{\partial F_2}{\partial y} \right] + \left[F_3 \frac{\partial f}{\partial z} + f\,\frac{\partial F_3}{\partial z} \right] \\ &\equiv (\nabla f) \cdot \underline{\mathbf{F}} + f\,(\nabla \cdot \underline{\mathbf{F}})\,. \end{aligned}$$

Similarly (3) and (1) $\implies$

$$\nabla \times (f\,\underline{\mathbf{F}}) \equiv (\nabla f) \times \underline{\mathbf{F}} + f\,(\nabla \times \underline{\mathbf{F}})\,.$$

We introduce the **LAPLACIAN** operator ∇^2 : scalar field $\to$ scalar field, defined by

$$\nabla^2 f \equiv \nabla \cdot (\nabla f) \equiv \nabla \cdot \left(\frac{\partial f}{\partial x}\underline{\mathbf{i}} + \frac{\partial f}{\partial y}\underline{\mathbf{j}} + \frac{\partial f}{\partial z}\underline{\mathbf{k}} \right) \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\,.$$

It is simple matter to show using the definitions (1), (2) and (3) that

$$\nabla \times (\nabla f) = \underline{0} \qquad \text{and} \qquad \nabla \cdot (\nabla \times \underline{\mathbf{F}}) = 0\,.$$

RECALL: Conservative Vector Fields and Path Independence

Given a vector field $\underline{\mathbf{F}}(\underline{\mathbf{r}}) \equiv F_1(\underline{\mathbf{r}})\underline{\mathbf{i}} + F_2(\underline{\mathbf{r}})\underline{\mathbf{j}} + F_3(\underline{\mathbf{r}})\underline{\mathbf{k}}$, if there exists a $\phi(\underline{\mathbf{r}})$ satisfying

$$\frac{\partial \phi}{\partial x} = F_1\,, \qquad \frac{\partial \phi}{\partial y} = F_2\,, \qquad \frac{\partial \phi}{\partial z} = F_3\,; \tag{4a}$$

then

$$\int_{C_{\underline{\mathbf{A}} \to \underline{\mathbf{B}}}} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}} = \int_{C_{\underline{\mathbf{A}} \to \underline{\mathbf{B}}}} \left[\frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy + \frac{\partial \phi}{\partial z}\,dz \right] = \int_{C_{\underline{\mathbf{A}} \to \underline{\mathbf{B}}}} d\phi = \phi(\underline{\mathbf{B}}) - \phi(\underline{\mathbf{A}}); \tag{4b}$$

that is, $\int_{C_{\underline{\mathbf{A}} \to \underline{\mathbf{B}}}} \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}}$ depends only on the start point $\underline{\mathbf{A}}$ and the end point $\underline{\mathbf{B}}$, NOT on the particular path C joining $\underline{\mathbf{A}}$ to $\underline{\mathbf{B}}$. We showed that a ϕ satisfying (4a) exists, and hence that (4b) holds, if and only if $\underline{\mathbf{F}}(\underline{\mathbf{r}})$ satisfies the 3 conditions

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}\,, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}\,, \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}\,. \tag{5}$$

Such an $\underline{\mathbf{F}}$ is called **CONSERVATIVE** with the corresponding ϕ being the **POTENTIAL** of $\underline{\mathbf{F}}$.
In terms of grad and curl, (5) $\equiv \nabla \times \underline{\mathbf{F}} = \underline{0}$ and (4a) $\equiv \underline{\mathbf{F}} = \nabla \phi$. Therefore we have that

$$\boxed{\underline{\mathbf{F}} \text{ conservative} \iff \nabla \times \underline{\mathbf{F}} = \underline{0} \iff \underline{\mathbf{F}} = \nabla \phi \text{ for some } \phi \iff (4b)\,.}$$