The Saffman-Taylor instability for a planar interface

Question 2 on Problem sheet 1 is known as the Saffman-Taylor instability. As well as for flow through a porous medium as discussed in the question, it also occurs in **Hele-Shaw cells,** when viscous fluid flows in a thin gap between two planar walls. In either case, the velocity $\mathbf{u}(x, y)$ is given by

$$\mathbf{u} = \nabla\phi \quad \text{and} \quad \phi = -p/\sigma, \tag{1}$$

for some constant σ where p is the fluid pressure. We analyse here the fingering instability which is observed when a less-viscous fluid advances under a pressure gradient into a more viscous fluid. For simplicity suppose that one of the fluids is dynamically negligible, so that $\phi = 0$ and $p = p_\infty$ in one fluid. We will however allow for surface tension between the two fluids. Let $y < Vt$ be filled with water and $y > Vt$ with air.

Basic state

Suppose first that the interface at position $y = Vt$ is planar. Then the basic state is

$$\begin{array}{rl} \text{Water} & \text{Air} \\ \phi_0 = Vy + const & \\ p_0 = -\sigma V(y - Vt) + p_\infty & p = p_\infty \end{array}$$

Now imagine that the interface suffers an infinitesimal perturbation having wave number k and amplitude ε. We anticipate that this perturbation will grow or decay exponentially in time, and check this assumption later. In that case the position of the interface becomes

$$y = Vt + \varepsilon e^{ikx+st} \qquad \text{where} \quad k > 0 \qquad (2),$$

where ε is arbitrarily small. At leading order in ε, all the perturbation quantities inherit this dependence on y and t, so that the velocity potential in the water becomes

$$\phi = \phi_0 + \varepsilon f(y) e^{ikx+st}.$$

Now $\nabla^2\phi = 0$, and thus the solution that decays as $y \to -\infty$ is

$$\phi = Vy + const + \varepsilon A e^{ikx+st+k(y-Vt)}, \tag{3}$$

for some constant A. The corresponding pressure is

$$p = p_0 + p_1 \quad \text{where} \quad p_1 = -\sigma\varepsilon A e^{ikx+st+k(y-Vt)}. \tag{4}$$

We therefore have:

Water	Air
$\phi = \phi_0 + \phi_1$	
$p = p_0 + p_1$	$p = p_\infty$

Our aim is now to find s.

If a **surface tension** γ acts between the fluids then the interface curvature at leading order is just $\partial^2 y/\partial x^2$ so that

$$[p] = \gamma \partial^2 y/\partial x^2 = -k^2 \gamma \varepsilon e^{ikx+st}, \tag{5}$$

where the jump is taken across the position of the perturbed interface. At leading order in ε this gives (check!)

$$p_\infty - \Big[-\sigma V \varepsilon e^{ikx+st} + p_\infty - \varepsilon\sigma A e^{ikx+st} \Big] = -k^2 \gamma \varepsilon e^{ikx+st}. \tag{6}$$

The first term here is the pressure in the air, the second, that in the water. Simplifying we find

$$A = -V - \gamma k^2. \tag{7}$$

Having determined the amplitude of the velocity perturbation, A, we can now calculate the velocity of the interface as $\partial\phi/\partial y$, which at leading order may again be evaluated at $y = Vt$, and this must correspond to $\partial y/\partial t = V + \varepsilon e^{ikx+st}$ (the **kinematic** condition.) This gives finally

$$s = Ak = -Vk \Big[1 + \frac{\gamma k^2}{V} \Big]. \tag{8}$$

We note that the dependence on t cancels out, justifying our assumption of an exponential time dependence above.

In the absence of surface tension, we see that $s > 0$ whenever $V < 0$, thus the interface perturbation grows (according to linear theory) if the air moves into the water. It decays, on the other hand, if water moves into air. In the unstable case, the fastest growing modes are those short waves for which $k \to \infty$.

If $\gamma > 0$ then surface tension is predicted to stabilise the shortest waves. The interface is still unstable, but the fastest growing mode now has a finite value of k and s.