Hydrodynamic Stability: 3: Circular Flow including 2007 Coursework

In terms of cylindrical polar coordinates $(r,\, \theta,\, z)$, **circular flow** has a velocity of the form $\mathbf{u} = (0,\, V(r),\, 0)$. This satisfies the inviscid equations (0.6) for any function $V(r)$, but if the viscous terms are included V must be of the form $V = Ar + B/r$. This we recognise as a solid body rotation with angular speed A, and a line vortex at $r = 0$ of strength $2\pi B$.

Suppose circular flow takes place between the two rigid cylinders $r = R_1$ and $r = R_2$, where $R_1 < R_2$. An inviscid stability analysis perturbs the flow generally as

$$\mathbf{u} = (0,\, V(r),\, 0) + \varepsilon[u_r'(r),\, u_\theta'(r),\, u_z'(r)]\zeta \qquad \text{where} \quad \zeta = e^{ikz+im\theta+st} \tag{3.1}$$

and $0 < \varepsilon \ll 1$. The boundary conditions require $u_r' = 0$ on $r = R_1, R_2$. We will only consider 2-D and axisymmetric perturbations so that respectively either $k = 0$ or $m = 0$.

Two-dimensional perturbations: Setting $k = 0$, $u_z' = 0$ and introducing a stream-function $\psi_1 = \psi(r)\zeta$, so that $u_r' = im\psi\zeta/r$ and $u_\theta' = -\psi'\zeta$, we find

$$\left(s + im\frac{V}{r}\right)\nabla^2\psi - \frac{1}{r}\frac{dQ}{dr}im\psi = 0 \qquad \text{where} \quad Q = \frac{1}{r}\frac{d}{dr}(rV)\ , \tag{3.2}$$

and $\nabla^2\psi = \psi'' + \psi'/r - m^2\psi/r^2$. Here Q is the z-component of the vorticity of the circular flow. Note that it is constant for the constant rotation $V = Ar$ and zero (except on $r = 0$) for the line vortex $V = B/r$.

(1.) Assuming dQ/dr is continuous, prove that a necessary condition for instability is that $dQ/dr = 0$ somewhere in the range $R_1 < r < R_1$.
(You may find it helpful to write $s = -im\Omega$ where Ω is a complex constant, and modify the proof of Rayleigh's inflection point theorem.)

We now consider **inviscid axisymmetric perturbations**, so that $m = 0$ (but $k \neq 0$).

(2.) Show that axisymmetric perturbations must satisfy

$$\frac{d^2u_r'}{dr^2} + \frac{1}{r}\frac{du_r'}{dr} - \frac{u_r'}{r^2} - k^2u_r' - \frac{k^2}{s^2}\Phi u_r' = 0 \qquad \text{where} \quad \Phi = \frac{1}{r^3}\frac{d}{dr}(r^2V^2)\ . \tag{3.3}$$

This equation can be written in *Sturm-Liouville* form

$$\frac{d}{dr}\left(r\frac{du_r'}{dr}\right) - \left(k^2r + \frac{1}{r}\right)u_r' + \lambda(-r\Phi)u_r' = 0 \qquad \text{in} \quad R_1 < r < R_2 \tag{3.4}$$

with $u_r' = 0$ on $r = R_1$ and $r = R_2$, where the eigenvalue $\lambda = k^2/s^2$. Sturm-Liouville theory tells us that (3.4) has infinitely many eigenvalues, λ, that they are all real, and they are positive if $\Phi < 0$ but negative if $\Phi > 0$. Positive λ corresponds to real s and **instability.** This agrees with the physical argument presented in lectures showing that if $\Phi < 0$ then energy is released by interchanging two fluid rings while conserving their angular momentum. Note that the smallest λ corresponds to the largest growth rate, s.

Numerical Solution

For given $V(r)$, R_1 and R_2, equation (3.3) can be solved numerically to find $s(k)$. Unlike for most hydrodynamic stability problems, we have shown that s^2 is real and the calculation is simplified. There are "black box" routines for solving Sturm-Liouville problems, such as the NAG Fortran routine D02KAF. (For documentation see the web page http://www.nag.co.uk/numeric/FL/manual/pdf/D02/d02kaf.pdf) You could alternatively write the problem in matrix form and use LAPACK or use a boundary value solver in say MATLAB. Or you could use the "Rayleigh-Ritz" method descibed on problem sheet 2.

Set $R_2 = 1$. If your birthday is the α'th day of the β'th month, set $\Omega_1 = 3\beta + \alpha$ and $\Omega_2 = \beta$. For your $V(r)$ take the steady circular **viscous** flow $V = Ar + B/r$ which might be expected when $V = R_1\Omega_1$ on $r = R_1$ and $V = R_2\Omega_2$ on $r = R_2$. Now solve the stability problem (3.4) for this flow numerically, treating k and R_1 as parameters.

(3.) Find the value of R_1 for which $\Phi = 0$. Choose a value for R_1 greater than this value. Write a program to calculate s as a function of k for this value of R_1 and plot the results. What is the maximum growth rate, s_{max}, of s for your $V(r)$? Does it occur at small, large or intermediate values of the wavenumber k?

(4.) Now consider other values of R_1. Determine how s_{max} behaves in the narrow gap limit as R_1 increases towards R_2, keeping $R_2 = 1$.

The coursework for this course consists of the 4 questions in bold, surrounded by boxes above. This should be handed in by 11^{th} May to the Aeronautics office. The numerical parts (3 & 4) will be given higher priority in the marking. Be sure to give your birthdate on your coursework.

The effect of viscosity

The Rayleigh Circulation Criterion ($\Phi > 0$) is a good indicator of the stability of these flows, even when the effects of viscosity are included. For the flow $V(r) = Ar + B/r$, it is appropriate to use a parameter known as the **Taylor number**,

$$T = \frac{4AB\rho^2 R_1^2}{\mu^2}; \qquad \text{instability if} \quad T \succeq 1700. \tag{3.5}$$

The effect of viscosity is stabilising for this problem – those flows regarded as stable by inviscid theory are indeed stable, whereas some flows regarded as unstable may in fact be stable until the Taylor number is large enough. This is especially true if the cylinders rotate in opposite directions when the inviscid theory always predicts instability.