Hydrodynamics Stability 4: The e^n method for 2-D incompressible flow.

The Orr-Somerfeld Equation (OSE) is exact only for genuinely unidirectional flows, which for viscous flow requires $u(y) = ay^2+by+c$. However, at high Reynolds number flow in the boundary layer is almost parallel, in that $|u| \gg |v|$. Boundary layer instability has great practical importance, as it triggers the **transition** to turbulence on a wing, with great consequences for separation and resultant drag. It is thus tempting, and mathematically justifiable as $R_e \to \infty$, to apply the OSE for boundary layer profiles, assuming the flow is exactly parallel with whatever streamwise velocity the boundary layer equations predict. Unfortunately, we are then most interested in the critical Reynolds number, R_c, a relatively small value of R_e where the parallel flow assumption of the OSE is questionable. While results so obtained usually agree reasonably with experiments, the arguments are not strictly valid.

A further limitation of the linear OSE is that it cannot predict where breakdown of the laminar boundary layer will occur. For self-similar profiles, one can argue that the local Reynolds number is x-dependent (proportional to $x^{1/2}$ for the Blasius layer on a flat plate, for example) and thus the critical value R_c can be associated with a critical position x_c at which instability commences. However, the linear theory assumes the perturbation is infinitesimal, and as it grows exponentially it is advected downstream. At some stage the linear theory will break down but the flow will still be laminar. Fairly soon thenafter it will become turbulent, but some criterion is required to determine where this can be expected.

A simple criterion, known as the "e^n method", is widely used and fairly successful. Essentially this states that transition occurs roughly when linear theory predicts that an initial disturbance will have grown by a factor of e^n. A common choice is $n = 9$, for which $e^n \simeq 8103$, but the optimum choice of n varies between applications.

For the temporal approach we have used almost exclusively so far, a disturbance has amplitude $A = A_0 e^{ikx+(s_r+is_i)t]}$, that is the spatial Fourier mode e^{ikx} is assumed to grow in time like $e^{s_r t}$. However, as the disturbance moves downstream, the local boundary layer velocity changes so that s_r should not be treated as constant. Between two times t_2 and t_1 the corresponding disturbance amplitudes A_1 and A_2 are related by

$$\left|\frac{A_2}{A_1}\right| = \exp\left[\int_{t_1}^{t_2} s_r(t)\,dt\right] \tag{4.1}$$

More useful in this context is a representation in **spatial modes** with typical amplitude $A = A_0 e^{i\omega t+i(k_r+ik_i)x}$ which describes how a wave of fixed (real) frequency ω grows or decays downstream like $e^{-k_i x}$. Once more, as the base flow evolves k_i should be regarded as variable and the amplitude ratio at two positions x_1 and x_2 is

$$\left|\frac{A_2}{A_1}\right| = \exp\left[-\int_{x_1}^{x_2} k_i(x)dx\right]. \tag{4.2}$$

These two approaches are essentially equivalent. If we regard both k and s as complex, so that $k = k_r + ik_i$ and $s = s_r + is_i$, then the eigenvalue equation (or **dispersion relation**) can be represented by

$$\mathcal{F}(k,s) = 0 \qquad \text{for some function } \mathcal{F}. \tag{4.3}$$

Assuming that (4.3) defines an analytic relation between the complex variables k and s we can use the Cauchy-Riemann relations

$$\frac{\partial s_r}{\partial k_r} = \frac{\partial s_i}{\partial k_i} \qquad \frac{\partial s_r}{\partial k_i} = -\frac{\partial s_i}{\partial k_r} \ .$$

Close to the neutral curve, where $k \simeq k_r$ and $s \simeq i\omega$ we have therefore approximately

$$s_r \simeq -\left(\frac{\partial \omega}{\partial k}\right) k_i \ . \tag{4.4}$$

This relation can be used to relate (4.2) and (4.1) although strictly (4.4) is only valid for small growth rates.

Using the e^n method

For a given boundary layer flow, we solve the OSE for a range of real k and downstream velocity profiles. We then convert to the spatial mode formulation using (4.4).

Consider the characteristic neutral curve in (x, ω) space as in the figure. A wave of fixed frequency ω propagating downstream from position x_0 is initially damped, but may enter the unstable region at some value x_1 and exiting at a value x_2. If the growth defined by (4.2) is greater than say e^9 we can expect transition to occur.

More precisely, at a fixed position x, we consider the growths over all frequencies ω, and choose the maximum, that is we define

$$N(x) = \max_{\omega} \left[\log |A(x,\omega)/A_0|\right] \ .$$

We then find the smallest value of x such that $N(x) = 9$. This gives our estimate for the transition point. If we superimpose all the $\log |A/A_0|$ curves for different ω on the same graph, the function $N(x)$ appears as the "envelope" of these curves.

The e^n method has a number of limitations. For example, as presented above, it assumes that A_0 is the same for all frequencies ω. In reality some frequencies will be excited more than others. The question of boundary layer 'receptivity', i.e. how the boundary layer responds to external forcing has not been considered. It is highly questionable whether the non-linear regime can be adequately represented by the exponential growth mechanism.

Nevertheless, agreement with 2-D incompressible flows is reasonable. The method can also be used for modes with a cross-flow component and for compressible flows, but we will not consider this.