Hydrodynamic Stability 2004

You may attempt all questions. Full marks can be obtained for complete answers to three questions. The use of lecture notes is NOT allowed

1. Two fluids with the same density ρ flow in $y > 0$, with uniform velocity $(U, 0, 0)$, and in $y < 0$ with velocity $(-U, 0, 0)$. The interface is perturbed in the form $y = \varepsilon\zeta$ where $\zeta = e^{ikx+st}$ and $0 < \varepsilon \ll 1$.

Show that the perturbed velocity in $y > 0$ is

$$\mathbf{u} = (U, 0, 0) + \varepsilon\nabla\phi_1 \qquad \text{where} \quad \phi_1 = -(s + ikU)\zeta e^{-ky}/k$$

and find the corresponding velocity potential ϕ_2 in $y < 0$. Show that the growth rate s is given by

$$s^2 = k^2U^2.$$

Discuss briefly the physical significance of this result. Would you expect to see perturbations of long or short wavelengths emerging? How would the inclusion of surface tension or viscosity influence the most unstable wavelength?

[*You may assume the flow is irrotational,* $\mathbf{u} = \nabla\phi$, *and that it obeys the time dependent Bernoulli equation,*

$$p + \tfrac{1}{2}\rho|\mathbf{u}|^2 + \rho\frac{\partial\phi}{\partial t} = \text{constant.}]$$

2. Derive the Rayleigh equation for the perturbed 1-D profile in $y_1 < y < y_2$

$$\mathbf{u} = (U(y),\, 0,\, 0) + \varepsilon(\phi'(y),\, -ik\phi(y),\, 0)e^{ik(x-ct)},$$

where $U(y)$ is a given smooth function, in the form

$$(U-c)(\phi'' - k^2\phi) - \phi U'' = 0 \qquad \text{with} \quad \phi(y_1) = 0 = \phi(y_2).$$

Prove that

$$\int_{y_1}^{y_2} \left(|\phi'|^2 + k^2|\phi|^2 + \frac{(U-c^*)|\phi|^2 U''}{|U-c|^2} \right) dy = 0,$$

where c^* is the complex conjugate of c. Assuming the base flow is inviscidly unstable deduce that it is necessary that

(i) $U'' = 0$ for some value of y, and

(ii) $(U - U_s)U'' < 0$ for some value of y, where $U(y) = U_s$ at the point where $U'' = 0$.

Sketch the following flow profiles in $-1 < y < 1$. Which might be inviscidly unstable?

(a) $U = \tanh(y)$, (b) $U = y^3$, (c) $U = 1 + y^2$?

3. Write the skeleton of a computer program, in a language of your choice using any solution method, to solve the Rayleigh equation

$$(U - c)(\phi'' - k^2\phi) - U''\phi = 0 \qquad \text{with} \quad \phi(0) = 0 = \phi(1)$$

for various wave-numbers k and a given smooth function $U(y)$ in $0 < y < 1$. You may assume the existence of subroutines

(a) EVALSOLVE (A, B, N, Lambda) which finds all the (complex) eigenvalues λ for $n \times n$ matrices A and B satisfying $A\mathbf{x} + \lambda B\mathbf{x} = 0$. It returns the eigenvalues as a complex vector Lambda.

(b) ODESOLVE (f, t0, t1, x0, x1) which solves the simultaneous first order differential equations for $\mathbf{x}(t)$ in $t_0 < t < t_1$

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

for given functions $\mathbf{f}$ and initial values $\mathbf{x}_0$. It returns the values $\mathbf{x}_1 = \mathbf{x}(t_1)$.

(c) FINDROOT (g, x0, x1, root1, root2) which attempts to find a complex zero of a function $g(x)$ near $x = x_0 + ix_1$. It returns the root as (root1 + i root2).

Give a reasonable amount of detail, but concentrate on the solution algorithm rather than facets of the particular computer language; make sure you define the matrices A and B and/or the functions $\mathbf{f}$ and g clearly.

4. In cylindrical polar coordinates $(r,\,\theta,\,z)$ the governing inviscid equations are

$$\left.\begin{aligned}
\rho\left(\frac{Du_r}{Dt}-\frac{u_\theta^2}{r}\right) &= -\frac{\partial p}{\partial r}\\
\rho\left(\frac{Du_\theta}{Dt}+\frac{u_r u_\theta}{r}\right) &= -\frac{1}{r}\frac{\partial p}{\partial \theta}\\
\rho\frac{Du_z}{Dt} &= -\frac{\partial p}{\partial z}\\
\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z} &= 0
\end{aligned}\right\}\qquad \frac{Df}{Dt}=\frac{\partial f}{\partial t}+u_r\frac{\partial f}{\partial r}+u_\theta\frac{1}{r}\frac{\partial f}{\partial \theta}+u_z\frac{\partial f}{\partial z}\ .$$

Circular flow between two cylinders is perturbed axisymmetrically in the form

$$\mathbf{u}=(0,\,V(r),\,0)+\varepsilon[u_r'(r),\,u_\theta'(r),\,u_z'(r)]\zeta \qquad \text{where} \quad \zeta=e^{ikz+st}\ .$$

Show that the perturbation must satisfy

$$\frac{d^2u_r'}{dr^2}+\frac{1}{r}\frac{du_r'}{dr}-\frac{u_r'}{r^2}-k^2u_r'-\frac{k^2}{s^2}\Phi u_r'=0 \qquad \text{where} \quad \Phi=\frac{1}{r^3}\frac{d}{dr}(r^2V^2)\ .$$

A circular bearing is modelled by a cylinder $r=R_1$ rotating with angular speed Ω_1 inside a stationary cylinder $r=R_2$ with $R_2>R_1$. The steady *viscous* flow is of the form $V=Ar+B/r$. Find the constants A and B for this flow and find Φ.

Given that for fixed k the eigenvalues k^2/s^2 can be positive only if $\Phi<0$ somewhere, discuss whether the flow is likely to be stable.