1. A fluid with density ρ_1 occupies $y > 0$, while $y < 0$ contains fluid of density ρ_2. Both fluids are initially at rest, and the gravitational acceleration is $(0, -g, 0)$. The interface is perturbed in the form $y = \varepsilon\zeta$ where $\zeta = e^{ikx+st}$ and $0 < \varepsilon \ll 1$.

Show that the perturbed velocity in $y > \varepsilon\zeta$ is

$$\mathbf{u} = \varepsilon\nabla\phi_1 \qquad \text{where} \quad \phi_1 = -s\zeta e^{-ky}/k$$

and find the corresponding velocity potential ϕ_2 in $y < \varepsilon\zeta$. Show that the growth rate s is given by

$$s^2 = kg\,\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}.$$

Discuss briefly the physical significance of this result. Would you expect to see perturbations of long or short wavelengths emerging? How would the inclusion of surface tension or viscosity influence the most unstable wavelength?

Use the above result to estimate the phase speed ω/k of a surface wave on a deep lake, given by $y = \varepsilon\cos(kx - \omega t)$, when the wavelength is 1 metre.

[*You may assume that the flow is irrotational,* $\mathbf{u} = \nabla\phi$, *and that it obeys the time dependent Bernoulli equation,*

$$p + \tfrac{1}{2}\rho|\mathbf{u}|^2 + \rho\frac{\partial\phi}{\partial t} + \rho g y = \text{constant.}]$$

2. The perturbed 1-D profile in $y_1 < y < y_2$

$$\mathbf{u} = (U(y),\, 0,\, 0) + \varepsilon(\phi'(y),\, -ik\phi(y),\, 0)e^{ik(x-ct)},$$

gives rise to the Orr-Sommerfeld equation (OSE),

$$(U - c)(\phi'' - k^2\phi) - \phi U'' = \frac{1}{ikR_e}\left(\phi'''' - 2k^2\phi'' + k^4\phi\right) \tag{1}$$

with the boundary conditions

$$\phi(y_1) = 0 = \phi(y_2), \quad \phi'(y_1) = 0 = \phi'(y_2). \tag{2}$$

Describe clearly, but in not too much detail, how to reduce the problem (1) and (2) to the matrix eigenvalue problem

$$A\mathbf{x} + cB\mathbf{x} = 0,$$

where N is a suitable integer, A and B are complex $N \times N$ matrices and $\mathbf{x}$ is an N-vector, using **either** a spectral method **or** finite differences. **[50%]**

In each of the following cases discuss the appropriateness of using the OSE, and any required modifications to it.

(a) $U = 1 - y^2$ for $-1 < y < 1$,
(b) $U = \tanh y$ for $-\infty < y < \infty$
(c) $U = y$ for $0 < y < 3$
(d) $U = (1 - r^2)$ for $r^2 = y^2 + z^2 < 1$.

Sketch, **qualitatively**, in the $(R_e,\, k)$-plane, with $k > 0$ and $R_e > 0$, the stability regions you would expect to obtain. Indicate clearly the stable and unstable regions in your diagrams and comment on any significant flow features. **[50%]**

3. An inviscid, unidirectional flow $(U(y), 0, 0)$ is perturbed by a three-dimensional mode so that

$$\mathbf{u} = (U(y),\, 0,\, 0) + \varepsilon\big(u(y),\, v(y),\, w(y)\big)e^{ikx+ilz+st}, \qquad p = p_0 + \varepsilon p_1(y)e^{ikx+ilz+st}$$

for real positive k and l, where ε is a small constant, $0 < \varepsilon \ll 1$. Starting from the dimensionless Navier-Stokes equations,

$$\frac{D\mathbf{u}}{Dt} = -\nabla p + R_e^{-1}\nabla^2\mathbf{u}, \qquad \nabla \cdot \mathbf{u} = 0$$

obtain the linearised equations for u, v and w.

Denoting d/dy by $'$ and defining

$$\overline{k} = (k^2 + l^2)^{1/2}, \qquad \overline{p} = \overline{k}p_1/k \qquad \overline{u} = (ku + lw)/\overline{k} \qquad \overline{s} = s\overline{k}/k, \qquad \overline{R} = kR_e/\overline{k}$$

show that

$$\left.\begin{aligned}(\overline{s} + i\overline{k}U)\overline{u} + vU' &= -i\overline{k}\overline{p} + \overline{R}^{-1}(\overline{u}'' - \overline{k}^2\overline{u})\\ (\overline{s} + i\overline{k}U)v &= -\overline{p}' + \overline{R}^{-1}(v'' - \overline{k}^2 v)\\ i\overline{k}\overline{u} + v' &= 0\end{aligned}\right\}$$

Explain carefully what can be deduced about the relative importance of instability to two-dimensional and three-dimensional modes for

(a) inviscid flow ($R_e = \infty$), and

(b) viscous flow.

4. A two-dimensional, laminar boundary layer has a known velocity component parallel to the wall $u(x, y)$, where x and y are measured parallel and normal to the wall respectively.

A friend supplies you with a computer program which solves the Orr-Sommerfeld equation calculating the complex wave speed c for fixed $U(y)$, R_e and real wave-number k, in the usual notation.

Discuss the use of the e^n-method for predicting the position, $x = x_T$, of the onset of transition in this boundary layer. You should include in your description the assumptions made at various stages in the argument, and the limitations and advantages of the method as a predictive tool.