Hydrodynamic Stability: Introductory Notes **A.J. Mestel** **January 2007**

We assume that at every point $\mathbf{x}$ of the fluid, and at all times t, we can define properties like density $\rho(\mathbf{x},\, t)$, velocity $\mathbf{u}(\mathbf{x},\, t)$, and pressure $p(\mathbf{x},\, t)$, and that these vary smoothly (differentiably) over the fluid. Note that we do not deal with the dynamics of individual molecules. A small volume δV thus has mass $\delta V \rho$ and momentum $\delta V \rho \mathbf{u}$.

The material derivative: A *fluid particle*, sometimes called a *material element*, is one that moves with the fluid, so that its velocity is $\mathbf{u}(\mathbf{x},\, t)$ and its position $\mathbf{x}(t)$ satisfies $\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x},\, t)$. The rate of change of a quantity as seen by a fluid particle is called the *material derivative* and written D/Dt. It is given by the chain rule as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla. \tag{0.1}$$

Mass conservation: $$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0. \tag{0.2}$$

For an *incompressible* fluid, the density of each material element is constant, and

Incompressible flow: $$\frac{D\rho}{Dt} = 0 \qquad \Longrightarrow \qquad \nabla\cdot\mathbf{u} = 0. \tag{0.3}$$

In this course we shall concentrate on fluids that are incompressible and have uniform density, so that ρ is independent of both $\mathbf{x}$ and t.

Streamfunctions in 2D and axisymmetry

For two-dimensional flows, the condition $\nabla \cdot \mathbf{u} = 0$ is automatically satisfied by

$$\mathbf{u} = \nabla \wedge \big(0, 0, \psi(x,y)\big) = (\psi_y, -\psi_x, 0). \tag{0.4}$$

$\psi(x,y)$ is called the *streamfunction.*

In axisymmetric flows, in terms of **cylindrical** polar coordinates $(r,\, \theta,\, z)$, the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ is satisfied using the **Stokes streamfunction**, $\psi(r,\, z)$,

$$\mathbf{u} = \nabla \wedge (0,\, \frac{\psi}{r},\, 0) = \left(-\frac{1}{r}\frac{\partial \psi}{\partial z},\, 0,\, \frac{1}{r}\frac{\partial \psi}{\partial r}\right) \;. \tag{0.5}$$

The Navier-Stokes Equations for an incompressible fluid

$$\rho\frac{D\mathbf{u}}{Dt} \equiv \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \mathbf{F} + \mu\nabla^2\mathbf{u} \;. \tag{0.6}$$

$$\nabla \cdot \mathbf{u} = 0 \;. \tag{0.7}$$

In (0.6) μ is the viscosity, assumed constant, and $\mathbf{F}$ a body force, perhaps gravity, $\mathbf{F} = \rho\mathbf{g}$. In cylindrical polar cordinates, $(r,\, \theta,\, z)$, with velocity $\mathbf{u} = (u_r,\, u_\theta,\, u_z)$, (0.6-0.7) become

$$
\begin{array}{llcl}
\textbf{Cylindrical} & \rho\left(\dfrac{Du_r}{Dt}-\dfrac{u_\theta^2}{r}\right) &=& -\dfrac{\partial p}{\partial r}+\mu\left(\nabla^2 u_r-\dfrac{u_r}{r^2}-\dfrac{2}{r^2}\dfrac{\partial u_\theta}{\partial \theta}\right)\\
\textbf{Polar} & \rho\left(\dfrac{Du_\theta}{Dt}+\dfrac{u_r u_\theta}{r}\right) &=& -\dfrac{1}{r}\dfrac{\partial p}{\partial \theta}+\mu\left(\nabla^2 u_\theta-\dfrac{u_\theta}{r^2}+\dfrac{2}{r^2}\dfrac{\partial u_r}{\partial \theta}\right)\\
\textbf{Coordinates} & \rho\dfrac{Du_z}{Dt} &=& -\dfrac{\partial p}{\partial z}+\mu\nabla^2 u_z\\
& \multicolumn{3}{c}{\dfrac{1}{r}\dfrac{\partial(ru_r)}{\partial r}+\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}+\dfrac{\partial u_z}{\partial z}=0}
\end{array}
\qquad (0.8)
$$

where

$$
\frac{Df}{Dt}=\frac{\partial f}{\partial t}+u_r\frac{\partial f}{\partial r}+u_\theta\frac{1}{r}\frac{\partial f}{\partial \theta}+u_z\frac{\partial f}{\partial z} \qquad \text{and} \qquad \nabla^2 f=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}+\frac{\partial^2 f}{\partial z^2}\ .
$$

Boundary Conditions

In order to determine the velocity $\mathbf{u}(\mathbf{x}, t)$ and pressure $p(\mathbf{x}, t)$ in some region V, we need to know what boundary conditions to apply on the surface S. The appropriate conditions to apply are that the velocity and the total stress should be continuous across any interface. Here 'total stress' includes any **surface tension** (see below.)

(a) Fluid/solid boundaries: A solid boundary can provide whatever stress is needed to support the fluid motion, so it is sufficient to require that the fluid velocity $\mathbf{u}$ be the same as the velocity of the boundary. Thus for a stationary boundary

$$
\mathbf{u}=0\ . \qquad (0.9)
$$

Note that (0.9) requires that the tangential velocity components be zero as well as the normal component. In **inviscid flow** only the normal velocity need be continuous at an interface, and a 'slip velocity' must be permitted. The presence in the Navier-Stokes equation of the second derivative $\mu\nabla^2\mathbf{u}$ requires an extra boundary condition.

(b) Fluid/fluid boundaries: These are more complicated, because the interface can move. Furthermore, it is a physical fact that an extra normal stress, due to **surface tension**, acts on the interface. This extra stress takes the form $\gamma K(\mathbf{x})$ where γ is the positive surface tension constant, and K is the curvature of the fluid surface, which can be defined by $K=\nabla\cdot\widehat{\mathbf{n}}$ where $\widehat{\mathbf{n}}$ is the unit normal to the interface.

If one of the fluids is dynamically negligible, as often happens with a liquid/gas interface, then we can treat one fluid as having a constant pressure p_0 and neglect its motion. If the interface is stationary, then the appropriate boundary conditions to apply on the other fluid are zero normal velocity and zero tangential stress. (So if the surface is $y=0$ and velocity $(u, v, 0)$ then we have $v=0$ and $\mu\,\partial u/\partial y=0$. For inviscid flow, $\mu=0$ and the tangential stress condition is trivial.) If the interface moves and we describe its position at time t by the function $\zeta(\mathbf{x}, t)=0$, then the **kinematic boundary condition** for the normal velocity can be written

$$
\frac{D\zeta}{Dt}=0. \qquad (0.10)
$$

Inviscid and high-Reynolds-number Flows

When written in terms of nondimensional variables, a parameter, R_e, known as the Reynolds number appears in the equations. R_e essentially measures the relative importance of the inertial to the viscous forces. and is defined by $R_e = \rho LU/\mu$ where L is a typical length-scale of the problem, and U a typical velocity magnitude.

At low values of R_e, it can be proved that only one steady solution of the Navier-Stokes equations exists, and that this flow is stable in the sense defined below. For high values of R_e, there are many examples where more than one stable, steady solution is known to exist. Flow instability is strongly linked with the existence of more than one solution.

When $R_e \gg 1$, it is tempting to neglect the viscous terms, setting $\mu = 0$. If this is done, one of the boundary conditions must be omitted, usually allowing tangential slip. Some caution is necessary, as viscous **boundary layers** form near solid surfaces in which the velocity develops strong gradients so that the viscous term cannot be neglected. Boundary layers typically have thickness $L/R_e^{1/2}$ and must remain thin for the "core/layer" structure to be valid. Inside a steady boundary layer, where x and y are measured parallel and normal to the boundary, the governing equations for $\mathbf{u} = (u, v, 0)$

$$\rho(uu_x + vu_y) = -p_x + \mu u_{yy} \qquad p_y = 0 \qquad u_x + v_y = 0 \tag{0.11}$$

These equations are **parabolic** which means they must be solved in the downstream direction. The pressure does not vary across the layer and is determined by the conditions at "$y = \infty$" which means the external potential flow. The boundary layer equations tend to be valid so long as the pressure gradient is **favourable**, which means $-p_x > 0$. If the pressure gradient is unfavourable, there is a strong likelihood that **separation** of the boundary layer will occur. This is manifested by the solution to the boundary layer equations developing a singularity. Separation completely alters the external flow, and leads for example to "stall" of aircraft.

The **vorticity equation** is obtained by taking the curl of (0.6). Writing $\boldsymbol{\omega} = \nabla \wedge \mathbf{u}$ we have

$$\rho\left(\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\nabla\mathbf{u}\right) = \mu\nabla^2\boldsymbol{\omega}\ . \tag{0.12}$$

For two-dimensional flow, if we write $\mathbf{u} = \nabla \wedge (0,\, 0,\, \psi(x,\, y,\, t))$ and $\boldsymbol{\omega} = (0,\, 0,\, \omega)$ then

$$\rho\frac{D\omega}{Dt} = \mu\nabla^2\omega, \qquad \text{and} \quad \omega = -\nabla^2\psi\ . \tag{0.13}$$

A flow for which $\boldsymbol{\omega} = 0$ everywhere is said to be **irrotational**. Then we can introduce a velocity potential, ϕ, such that $\mathbf{u} = \nabla\phi$.

Inviscid Flows: As there is no source term in (0.12), vorticity can only be generated at boundaries. If $\mu = 0$ then a flow which is irrotational initially remains irrotational for all time. The **time-dependent Bernoulli theorem** states that for irrotational flows,

$$\frac{\partial\phi}{\partial t} + \tfrac{1}{2}|\nabla\phi|^2 + gy + \frac{p}{\rho} = \text{constant}. \tag{0.14}$$

Turbulence: At high values of R_e it is found experimentally that fluid flows tend to become unsteady and highly chaotic, even though a simple steady flow could exist in theory. Turbulent flows are difficult to analyse and have important practical implications. The manner in which **transition to turbulence** of a **laminar** flow occurs is an important topic. The first stage in this process is that the underlying steady flow becomes unstable. In this course we examine **Hydrodynamic Stability**.

Stability Concepts

For a given problem, we solve the governing equations and obtain a solution which we assume is steady, $\mathbf{u} = \mathbf{U}(\mathbf{x})$ with a corresponding pressure distribution $p = P(\mathbf{x})$. We then make a small perturbation to the flow, so that

$$\mathbf{u} = \mathbf{U}(\mathbf{x}) + \varepsilon\mathbf{u}'(\mathbf{x},\, t), \qquad p = P(\mathbf{x}) + \varepsilon p'(\mathbf{x},\, t) \tag{0.15}$$

where ε is a small positive constant. We then consider the behaviour of $\mathbf{u}'$. If $\varepsilon\mathbf{u}'$ remains small for all time, we say that the underlying flow is **stable**, whereas if it eventually becomes large no matter how small ε is, we say the flow is **unstable.**

The exact equations for $\mathbf{u}'$ and p' are

$$\rho\left(\frac{\partial\mathbf{u}'}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{u}' + \mathbf{u}'\cdot\nabla\mathbf{U} + \varepsilon\mathbf{u}'\cdot\nabla\mathbf{u}'\right) = -\nabla p' + \mu\nabla^2\mathbf{u}', \qquad \nabla\cdot\mathbf{u}' = 0 \tag{0.16}$$

Linear stability theory neglects the last term on the LHS, as ε is arbitrarily small. The resulting linear equation has solutions of the form $\mathbf{u}' = \widehat{\mathbf{u}}(\mathbf{x})e^{st}$ for some vector function $\widehat{\mathbf{u}}$ and constant s, and similarly $p' = \widehat{p}(\mathbf{x})e^{st}$. This is because none of the coefficients depends on t as $\mathbf{U}$ is steady. The general solution to this problem will be a linear combination of all these particular solutions. The possible values of s can be regarded as **eigenvalues** of the system. These can be real, but are in general complex

$$s = s_r + is_i, \qquad e^{st} = e^{s_r t}\left[\cos(s_i t) + i\sin(s_i t)\right]$$

(a) If for all possible values of s we have $s_r < 0$ we say the flow is **stable**.
(b) If there is at least one eigenvalue s for which $s_r > 0$, the flow is **unstable**.
(c) If $s_r = 0$ for some eigenvalue, we say the flow is **neutrally stable**. In this case nonlinear terms may be particularly important.

Surface stability: If the fluid has a free surface, this will deform in accordance with the normal stress associated with the perturbation velocity. Free surfaces can be unstable even at very low R_e.

The above approach looks at perturbation modes with a fixed spatial structure and examines how they evolve in time, a process known as **temporal stability.** An alternative approach, which is often appropriate, is to consider the spatial evolution of a localised disturbance in the flow. This disturbance may grow as it is advected downstream, so that the place where the instability occurs is far away from the disturbance. This is known as **convective instability.** In practice it is possible that the region of flow interest is too small for an instability of a given initial magnitude to develop. If a disturbance at a given position leads to growth at that position this is known as **absolute instability.**