Hydrodynamic Stability: Solutions to Problem Sheet 1 January 2007

1. In region 1, $\partial\phi/\partial y = V - \varepsilon k A_1 e^{ikx+st}$ on $y = Vt$ and so the kinematic condition is

$$0 = \frac{D}{Dt}\left(y - Vt - \varepsilon h_0 e^{ikx+st}\right) = -V - \varepsilon h_0 s e^{ikx+st} + V - \varepsilon k A_1 e^{ikx+st} + O(\varepsilon^2)$$

or

$$A_1 k = -h_0 s$$

In region 2, we write

$$\phi_2 = Vy + \varepsilon A_2 e^{ikx+(y-Vt)+st} \qquad \text{in} \quad y < Vt$$

and the kinematic condition gives $A_2 k = +h_0 s$. The dynamic condition is that the pressures should be equal, $p_1 = p_2$, on the interface. Now in fluid 1, the Bernoulli condition is

$$p_1 + \rho_1\left(\frac{\partial\phi_1}{\partial t} + \tfrac{1}{2}|\nabla\phi_1|^2 + gy\right) = \text{constant}$$

or

$$p_1 + \rho_1\left[\varepsilon A_1(s + kV)e^{\cdots}) + \tfrac{1}{2}(V - \varepsilon A_1 k e^{\cdots})^2 + g(Vt + \varepsilon h_0 e^{ikx+st})\right] = \text{constant}$$

or

$$p_1 + \tfrac{1}{2}\rho_1 V^2 + \rho_1 g V t + \varepsilon e^{ikx+st}\rho_1\left[(kV + s)A_1 - kVA_1 + gh_0\right] = \text{constant.}$$

In region 2 the same condition is

$$p_2 + \tfrac{1}{2}\rho_2 V^2 + \rho_2 g V t + \varepsilon e^{ikx+st}\rho_2\left[(-kV + s)A_2 + kVA_2 + gh_0\right] = \text{constant}$$

Combining these two, and imposing $p_1 = p_2$ on $y = Vt$, we have

$$\rho_1(sA_1 + gh_0) = \rho_2(sA_2 + gh_0)$$

or using the kinematic conditions we derived above, we get

$$s^2(\rho_1 + \rho_2) = gk(\rho_1 - \rho_2), \qquad \text{with instability if } \rho_2 > \rho_1.$$

We note that V has cancelled completely from the problem. If we work in a frame moving with constant velocity $(0,\ V,\ 0)$, the fluid equilibrium is stationary, and the stability condition we get is the same as in the Rayleigh-Taylor analysis (see (1.10)). We have done extra work for nothing, in this case! Formally, if we write $\mathbf{u} = \widehat{\mathbf{u}} + (0,\ V,\ 0)$ and $\widehat{y} = y - Vt$, then we can show that the Navier-Stokes equation for $\widehat{\mathbf{u}}(x, \widehat{y}, t$ is the same as that for $\mathbf{u}(x, y, t)$.

2. This problem is different however, as the Darcy equation is not "Galilean invariant" because it describes fluid motion through a stationary porous medium (rock, say). If we replace $\mathbf{u}$

by $\mathbf{u}+\mathbf{V}$, the equation changes. In the steady state, $\mathbf{u} = (0,\, V,\, 0)$ and in region 1, writing $\mathbf{u} = \nabla\phi_1$, we have

$$\begin{aligned} p_1 + \sigma_1\phi_1 + \rho_1 g y &= \text{constant}, \qquad \text{and} \\ p_2 + \sigma_2\phi_2 + \rho_2 g y &= \text{constant}, \qquad \text{in region 2.} \end{aligned}$$

In the unperturbed state, we could write $\phi_1 = \phi_2 = V(y - Vt)$ and if p_0 is the pressure at the interface,

$$\begin{aligned} p_1 &= p_0 - (\rho_1 g + \sigma_1 V)(y - Vt) \\ p_2 &= p_0 - (\rho_2 g + \sigma_2 V)(y - Vt) \end{aligned}$$

Now we perturb the interface as in question 1.

$$y = Vt + \varepsilon h_0 e^{ikx+st} \qquad \phi_1 = V(y - Vt) + \varepsilon A_1 e^{ikx - k(y-Vt)+st} \ .$$

The kinematic conditions are the same as in question 1, as they don't depend on the equation of motion, so that

$$A_1 k = -h_0 s, \qquad A_2 k = +h_0 s \ .$$

The dynamic condition $p_1 = p_2$ now takes the form on the interface

$$\begin{aligned} \sigma_1\phi_1 + \rho_1 g y &= \sigma_2\phi_2 + \rho_2 g y \\ \text{or} \qquad \sigma_1(V\varepsilon h_0 + \varepsilon A_1) + \rho_1 g h_0 &= \sigma_2(V\varepsilon h_0 + \varepsilon A_2) + \rho_2 g h_0 \\ \sigma_1(V - s/k) + \rho_1 g &= \sigma_2(V + s/k) + \rho_2 g \\ \text{or} \qquad \frac{s}{k}(\sigma_1 + \sigma_2) &= g(\rho_1 - \rho_2) + V(\sigma_1 - \sigma_2) \quad \text{as required.} \end{aligned}$$

Instability ($s > 0$) definitely occurs if $V(\sigma_1 - \sigma_2) \geqslant 0$ and $\rho_1 > \rho_2$, that is if heavy fluid is above light fluid. However, even if $\rho_2 > \rho_1$, instability will occur for large positive V if $\sigma_1 > \sigma_2$, or large negative V is $\sigma_2 > \sigma_1$.

3. Now we have $y = h(t)$ and the unperturbed interface moves with velocity $V(t) = h'(t)$, and $\phi = yh'(t)$. The Bernoulli relation is

$$p + \rho(yh''(t) + \tfrac{1}{2}V^2 + gy) = \text{constant} = p_0 + \rho(hh'' + \tfrac{1}{2}V^2 + gh)$$

where p_0 is the constant pressure on $y = h$. Thus $p = p_0 - \rho(y - h(t))(g + h'')$. Perturbing as in the question, the kinematic condition is

$$0 = \frac{D}{Dt}\big(y - h(t) - \varepsilon e^{ikx} f(t)\big) = -h' - \varepsilon e^{ikx} f' + V - k\varepsilon A e^{ikx} \qquad \Longrightarrow \qquad f' = -kA$$

The dynamic condition $p = p_0$ with $p + \rho(\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + gy$ constant requires on the surface

$$V'y + \varepsilon A' e^{ikx} + kh'\varepsilon A e^{ikx} + \tfrac{1}{2}(V - k\varepsilon A e^{ikx})^2 + g\varepsilon f e^{ikx} = \text{constant}$$

which reduces to

$$V'f + A' + gf = 0 \qquad \Longrightarrow \qquad f'' - k(V' + g)f = 0 \ .$$

Now if $V' + g$ is a positive constant, we see $f(t)$ will grow exponentially, giving instability, whereas if $V' + g$ is a negative constant f(t) will oscillate. If $V' + g$ is not constant, and changes sign, it is not obvious what will happen. But overall, we see that if the interface accelerates, it can overcome the gravitational instability.