Hydrodynamic Stability: Problem Sheet 1 January 2007

Stability of a moving interface: The three problems below refer to a scenario where fluid 1 with density ρ_1 occupies at $t = 0$ the region $y > 0$, while fluid 2 occupies $y < 0$. Gravity acts in the negative y-direction, and we neglect viscosity and surface tension. The interface moves in the y-direction with velocity $(0, V, 0)$, so that one fluid is attempting to flush out the other. Is this configuration stable? This is an important question in the cleaning out of fuel lines, and in oil recovery.

1. Assume V is constant and the flow is irrotational. Write the perturbed interface as

$$y = Vt + \varepsilon h_0 e^{ikx+st}$$

and the velocity potential in fluid 1 (and similarly in region 2) as

$$\phi_1 = Vy + \varepsilon A_1 e^{ikx-k(y-Vt)+st} \qquad \text{in} \quad y > Vt.$$

Use the kinematic condition to find A_1 (and A_2) and then use the dynamic condition to find s in terms of k. Deduce that the flow is stable if $\rho_1 < \rho_2$. Is this result obvious?

2. A technique used in oil wells is to force water into oil-rich rocks in an attempt to drive oil towards a region where it can be pumped out. As the rocky medium is porous, it should be modelled using the **Darcy equation** rather than the Navier-Stokes equation:

$$\sigma \mathbf{u} = -\nabla(p + \rho g y), \qquad \nabla \cdot \mathbf{u} = 0$$

Once again, take V as constant, and assume the positive constant $\sigma = \sigma_1$ in region 1, $\sigma = \sigma_2$ in region 2, as the porosity differs for water and oil.

Find the form of the pressure in equilibrium. Perturbing the surface as above, show that

$$(\sigma_1 + \sigma_2)(s/k) = (\sigma_1 - \sigma_2)V + g(\rho_1 - \rho_2)\ .$$

Deduce that instability can occur even if $\rho_2 > \rho_1$ in this case.

3. Finally, return to the inviscid Navier-Stokes and consider the case where the interface accelerates, so that $V = V(t) = h'(t)$. For simplicity take the case of a liquid above dynamically negligible air, so that $\rho_2 = 0$ and $p = p_0$, a constant, in the air. Find the unperturbed state, and show that $p = p_0 - \rho(g + V')(y - h)$ in the liquid $y > h(t)$. Now for $k > 0$, perturb the surface is the form

$$y = h(t) + \varepsilon e^{ikx} f(t) \qquad \text{with} \quad \phi = V(t)y + \varepsilon A(t) e^{ikx-k(y-h(t))}.$$

[Because the problem now has a time dependence, we cannot assume an exponential behaviour e^{st}.] Show from the kinematic condition that $f' = -kA$ and from the dynamic (Bernoulli) condition

$$f'' - kf(h'' + g) = 0 \qquad \text{in} \quad y > h(t)\ .$$

Deduce that a sufficiently large downwards acceleration can stabilise the interface. Try this in a bath some time!