Hydrodynamic Stability: Problem Sheet 2 February 2007

1. **Absolute and convective instability:** Consider the model problem for $u(x\,t)$

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} = \sigma u \qquad \text{where } V \text{ and } \sigma \text{ are constants.}$$

Consider perturbations to the solution $u = 0$ of the form $u = \varepsilon e^{ikx+st}$ and deduce the condition for stability.

Now consider the **initial value problem**, where $u = \varepsilon f(x)$ at $t = 0$. Show, by direct substitution or otherwise, that

$$u = \varepsilon e^{\sigma t} f(x - Vt) \ .$$

Does this solution agree with your stability criterion? Suppose $f(x) = \mathrm{sech}^2(x)$. Sketch $f(x)$.

Assuming $\sigma > 0$, look at the behaviour of u as $t \to \infty$ for **a fixed value of** x. If $u \to \infty$ the flow is **absolutely unstable**. If however, $u \to 0$ for fixed x, and yet the flow is unstable, it is **convectively unstable**. Deduce that for this perturbation the flow is absolutely unstable if $\sigma > 2V$, assuming $V > 0$.

2. Derive the Rayleigh equation for the perturbed 1-D profile

$$\mathbf{u} = (U(y),\,0,\,0) + \varepsilon(\psi'(y),\,-ik\psi(y),\,0)e^{ik(x-ct)} \ .$$

You may assume that at points where U or U' are discontinuous, the kinematic condition requires $\psi/(U - c)$ must be continuous, while the dynamic (pressure) condition requires that $(U - c)\psi' - U'\psi$ must be continuous.

Consider the piecewise linear flow

$$U = 0 \qquad \text{for} \quad |y| > 1, \qquad U = 1 - |y| \qquad \text{for} \quad |y| < 1 \ .$$

Sketch $U(y)$ and find $\psi(y)$ in $|y| > 1$ and $|y| < 1$ assuming that $\psi(y)$ is an even function. Show that

$$2k^2c^2 + k(1 - 2k - e^{-2k})c - [1 - k - (1 + k)e^{-2k}] = 0 \ .$$

Derive the condition for c to have an imaginary part, giving instability. [The resulting equation can only be solved numerically. The critical wave-number is $k = k_c \simeq 1.8$, with instability for $k < k_c$.]

3. Which of the following profiles in an unbounded fluid might be inviscidly unstable? (a) $U = \tanh(y)$, (b) $U = y^3$, (c) $U = e^{-y^2}$, (d) $U = y^4$, (e) $U = e^{-y^2}\cos(y)$?

4. Show that if c is an eigenvalue of the Rayleigh equation for a given $U(y)$ then so is its complex conjugate c^*. Is the same true of the Orr-Sommerfeld equation?

5. Write the skeleton of a computer program, in a language of your choice, to solve the Rayleigh equation for a given smooth function $U(y)$ in $0 < y < 1$. You may assume the existence of a subroutine which either

(a) finds the (complex) eigenvalues λ for $n \times n$ matrices A and B such that $A\mathbf{x} + \lambda B\mathbf{x} = 0$, or

(b) solves the simultaneous first order differential equations for $\mathbf{x}(t)$

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) \qquad \text{with} \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

for given functions $\mathbf{f}$ and initial values $\mathbf{x}_0$.

You may also assume the existence of a routine which attemps to find a zero of a function $g(x)$ in $x_0 < x < x_1$. Give a reasonable amount of detail; make sure you define the matrices A and B and the functions $\mathbf{f}$ and g.

6. Suppose that λ is an eigenvalue of the Sturm-Liouville equation for $y(x)$ in $a < x < b$

$$(p(x)y')' + q(x)y + \lambda w(x)y = 0 \qquad \text{with} \quad y(a) = 0, \quad y(b) = 0\ .$$

You may assume that the eigenvalues λ are real. Show that

$$\lambda = F[y] \equiv \frac{\int_a^b \left[p(y')^2 - qy^2\right]\, dx}{\int_a^b wy^2\, dx}\ .$$

It can be shown that the eigenvalues λ are **stationary values** of the functional $F[y]$; in particular, the smallest eigenvalue, λ_0 is the global minimum of $F[y]$ taken over all differentiable functions $y(x)$ which satisfy the boundary conditions. One can therefore obtain an estimate for λ_0 by substituting suitable functions y into $F[y]$ and taking the smallest result. This is called the **Rayleigh-Ritz** method. Before the advent of computers it was quite important.

Try the method for the simple problem in $0 < x < 1$

$$y'' + \lambda y = 0 \qquad \text{with} \quad y(0) = 0, \quad y(1) = 0$$

whose smallest eigenvalue is π^2. Use as a trial function for y a quadratic $y = x^2 + \alpha x + \beta$, choosing α and β to obey the boundary conditions. Evaluate $F[y]$ for this function and compare with the exact result.

If you wish, you may use the Rayleigh-Ritz method in your coursework, but you must be clear which trial functions you are using and should use more than one function.