Hydrodynamic Stability: Solutions to Problem Sheet 2 March 2007

1. If $u = \varepsilon e^{ikx+st}$, then $su + Viku = \sigma u$ or $s = \sigma - ikV$. For instability we need $\Re e[s] > 0$ or $\sigma > 0$.

If $u = \varepsilon e^{\sigma t} f(x - Vt)$, clearly $u = \varepsilon f(x)$ at $t = 0$. Substituting in the equation, the LHS is

$$\sigma\varepsilon e^{\sigma t} f(x - Vt) - V\varepsilon e^{\sigma t} f'(x - Vt) + V\varepsilon e^{\sigma t} f'(x - Vt) = \sigma u$$

the same as the RHS. Therefore it is the solution to the problem. Clearly the solution grows exponentially if $\sigma > 0$, just as derived before.

Now if $f(x) = \text{sech}\,^2 x$, then $f(\xi) \sim 4e^{-2|\xi|}$ as $\xi \to \pm\infty$. Thus as $t \to \infty$, assuming $V > 0$,

$$u \sim 4\varepsilon e^{\sigma t} e^{2x-2Vt} = 4\varepsilon e^{2x} e^{(\sigma-2V)t}$$

For constant x, this grows if $\sigma > 2V$. Thus the flow is **absolutely unstable** if $\sigma > 2V$, but **convectively unstable** is $2V > \sigma > 0$.

2. The first part is bookwork. For the given flow, $U'' = 0$ except for the points where U' is discontinuous. We should therefore solve the equation

$$\psi'' - k^2\psi = 0 \qquad \Longrightarrow \qquad \psi = \alpha e^{ky} + \beta e^{-ky}$$

separately in $y > 1$, $0 < y < 1$, $0 < y > -1$ and $y < -1$. We write

$$\psi = Ae^{-ky} \qquad \text{in } y > 1, \text{ and} \quad \psi = Be^{-ky} + Ce^{ky} \qquad \text{in } 0 < y < 1,$$

imposing $\psi \to 0$ as $y \to \infty$. Then since ψ is an even function, in $-1 < y < 0$ we must have $\psi = Be^{ky} + Ce^{-ky}$ and $\psi = Ae^{ky}$ in $-1 > y$. We now impose the kinematic and dynamic conditions at $y = 0$ and at $y = 1$. As U is continuous and the wavespeed c is constant, making $\psi/(U - c)$ continuous is equivalent to making ψ continuous. So we have

$$Ae^{-k} = Be^{-k} + Ce^{k}, \qquad \text{from continuity at } y = 1. \tag{1}$$

(We note ψ is continuous at $y = 0$.) Next we impose continuity of $(U - c)\psi' - U'\psi$ at $y = 0$. Evalauting the expression either side of $y = 0$, noting $U(0) = 1$ and $U'(0) = \pm 1$, we have

$$(1 - c)(-kB + kC) - (-1)(B + C) = (1 - c)(kB - kC) - (1)(B + C)$$

or

$$B(1 - k + ck) + C(1 + k - kc) = 0 \tag{2}$$

Similarly, on $y = 1$ we have $U = 0$ and $U' = 0$ or -1. Continuity of pressure then gives

$$-c(-kAe^{-k}) - 0 = (-c)(-kBe^{-k} + kCe^{k}) - (-1)(Be^{-k} + Ce^{k})$$

or

$$kcA = B(kc + 1) + Ce^{2k}(1 - kc)$$

Eliminating A from (1), we have

$$0 = B + Ce^{2k}(1-2kc) \qquad \text{which with (2) implies}$$

$$(1+k-kc) = e^{2k}(1-2kc)(1-k+ck)$$

or

$$2k^2c^2 + (k - ke^{-2k} - 2k^2)c + (1+k)e^{-2k} - (1-k) = 0$$

as required. Now this quadratic for c has real roots, giving stability, unless "$b^2 < 4ac$" or

$$k^2(1 - e^{-2k} - 2k)^2 < 8k^2[(1+k)e^{-2k} - (1-k)]$$

or

$$f(k) \equiv (1 - 2k - e^{-2k})^2 - 8[(1+k)e^{-2k} - (1-k)] < 0.$$

3. (a) $U'' = -2\text{sech}\,^2 x \tanh x = 0$ when $x = 0$. Now $U_s = U(0) = 0$, and $(U - U_s)U'' = -2\tanh^2 \text{sech}\,^2 x \leqslant 0$. So by Fjørtoft's theorem, the flow may be unstable.

(b) $U'' = 6y = 0$ when $y = 0$. But $U_s = 0$ and $(U - U_s)U'' = 6y^4 \geqslant 0$. So this flow is inviscidly stable.

(c) $U'' = e^{-y^2}(4y^2 - 2) = 0$ when $y^2 = 1/2$. So $U_s = e^{-1/2}$. Now $U'' > 0$ for $y^2 > 1/2$ and $U - U_s < 0$ for $y^2 > 1/2$. So $U''(U - U_s) < 0$ for $y^2 > 1/2$. Similarly $U''(U - U_s) < 0$ for $y^2 < 1/2$ and so flow may be unstable.

(d) $U'' = 12y^2 = 0$ when $y = 0$ and then $U = 0$. $U''(U - U_s) = 12y^6 \geqslant 0$ and so flow is stable.

(e) If $U = e^{-y^2}\cos y$ then $U'' = e^{-y^2}\big((4y^2 - 3)\cos y + 2y \sin y\big)$. Now when $y \gg 1$ $U \simeq 0$ and $U'' \simeq 4y^2 \cos y e^{-y^2}$ can be either positive or negative. So $(U - U_s)U'' \simeq U_s U''$ can be negative for suitable large y, so flow may be unstable.

4. We know that for some ψ, $(U - c)\psi'' - U''\psi = 0$ with $U = 0$ on the walls. Taking the complex conjugate of this equation, since $U(y)$ is real, we have

$$(U - c^*)(\psi^*)'' - U''(\psi^*) = 0$$

and ψ^* is zero on the walls. Thus c^* is also an eigenvalue, and the corresponding eigenfunction is ψ^*.

Now because the Orr-Sommerfeld equation has a factor ikR_e on the RHS, the equation changes when we take the conjugate. Thus ψ^* satisfies a different equation, and we cannot deduce that c^* is also an eigenvalue. in that case.

5. this is bookwork

6. If $y = x^2 + \alpha x + \beta$ and we make $y(0) = 0 = y(1)$ then $y = x^2 - x$. We have

$$F[y] = \frac{\int_0^1 (y')^2\,dy}{\int_0^1 y^2\,dy} = \frac{\int_0^1 (2x-1)^2\,dx}{\int_0^1 (x^2 - x)^2\,dx} = \frac{\frac{4}{3} - 2 + 1}{\frac{1}{5} - \frac{2}{4} + \frac{1}{3}} = \frac{1/3}{1/30} = 10.$$

This is not a bad approximation for $\pi^2 \simeq 9.87$.