Hydrodynamic Stability: 2. The Stability of Plane Parallel Flow

(Some of these notes derive from material provided by Dr Robert Hunt.)

Nondimensionalisation: Consider a basic 2D flow $(U_*(y_*), 0, 0)$ in the x-direction in an incompressible inviscid fluid between two plane boundaries $y_* = y_{1*}$ and y_{2*}. These boundaries may be either rigid (no normal velocity) or free (constant pressure), and either of them may be at infinity. The asterisk denotes a dimensional (physical) quantity; we nondimensionalize using a length L which is characteristic of the problem (e.g. $\frac{1}{2}(y_{2*} - y_{1*})$) and a velocity V (e.g. $\max|U_*(y_*)|$). Then defining

$$\mathbf{x} = \frac{1}{L}\mathbf{x}_*, \qquad \mathbf{u} = \frac{1}{V}\mathbf{u}_*, \qquad t = \frac{V}{L}t_*, \qquad p = \frac{1}{\rho V^2}p_* \quad \text{and} \quad U(y) = \frac{1}{V}U_*(y_*)$$

we obtain the momentum and continuity equations with $R_e = \rho VL/\mu$

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \left[R_e^{-1}\nabla^2\mathbf{u}\right], \qquad \nabla\cdot\mathbf{u} = 0. \tag{2.1}$$

To begin with we consider the inviscid limit, letting $R_e \to \infty$. Clearly, $\mathbf{u} = (U(y), 0, 0)$ is then a solution when $p = p_0$, a constant, for **any** profile $U(y)$. This is called a **unidirectional** or **parallel** flow. If we include viscosity, however, the only permissible functions $U(y)$ are quadratic, e.g. Poiseuille $U = 1 - y^2$ or simple shear $U = y$.

To analyse the stability, we try a small disturbance

$$\mathbf{u} = (U, 0, 0) + \varepsilon\mathbf{u}_1, \qquad p = p_0 + \varepsilon p_1, \tag{2.2}$$

where $\mathbf{u}_1 = (u_1, v_1, 0)$ is the disturbance velocity. (The analysis below can be performed for fully 3D disturbances with $\mathbf{u}_1 = (u_1, v_1, w_1)$ but it may be reduced to this 2D case using **Squires' theorem.**) Substituting these expressions into the equations of motion and linearising (i.e., ignoring $\mathbf{u}_1\cdot\nabla\mathbf{u}_1$) we obtain

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\mathbf{u}_1 + \left(v_1\frac{dU}{dy}, 0, 0\right) = -\nabla p_1, \qquad \nabla\cdot\mathbf{u}_1 = 0. \tag{2.3}$$

Let ψ_1 be the disturbance stream-function such that

$$u_1 = \frac{\partial\psi_1}{\partial y}, \qquad v_1 = -\frac{\partial\psi_1}{\partial x}. \tag{2.4}$$

We take double Fourier Transforms in x and t, i.e., we consider Fourier modes

$$\psi_1 = \widetilde{\psi}(k, y, \omega)e^{ikx - i\omega t}, \qquad p_1 = \widetilde{p}(k, y, \omega)e^{ikx - i\omega t}. \tag{2.5}$$

The equation of motion then gives

$$\left.\begin{aligned} (-i\omega + iUk)\frac{d\widetilde{\psi}}{dy} - ik\frac{dU}{dy}\widetilde{\psi} &= -ik\widetilde{p}, \\ -ik(-i\omega + iUk)\widetilde{\psi} &= -\frac{d\widetilde{p}}{dy} \end{aligned}\right\} \tag{2.6}$$

which lead to **Rayleigh's stability equation**

$$\left(U - \frac{\omega}{k}\right)\left(\frac{d^2\widetilde{\psi}}{dy^2} - k^2\widetilde{\psi}\right) - \frac{d^2U}{dy^2}\widetilde{\psi} = 0. \tag{2.7}$$

Writing ψ for $\widetilde{\psi}$, Rayleigh's equation is usually written

$$(U - c)(\psi'' - k^2\psi) - U''\psi = 0 \tag{2.8}$$

where $c = \omega/k$ is the phase speed and $\psi(y)$ is the "mode shape". c is in general complex and we write

$$c = c_r + ic_i \ . \tag{2.9}$$

For instability we need $c_i > 0$.

Rayleigh's equation must be solved subject to boundary conditions at $y = y_1$ and y_2; in the case of rigid boundaries, $\psi = 0$ there. This is an eigenvalue problem and will only have solutions for particular values of ω and k, leading to a dispersion relation $f(k, \omega) = 0$. Note that when k is real, if $\psi(y)$ is a solution corresponding to ω then ψ^* is also a solution corresponding to ω^*, where * denotes a complex conjugate.

We have assumed that the velocity profile $U(y)$ is twice differentiable in deriving (2.9). If this is not the case, for example if $U(y)$ or $U'(y)$ is discontinuous at some value, say $y = y_0$, we should solve (2.8) separately in $y < y_0$ and $y > y_0$, and then ensure that the pressure $\widetilde{p}$ is continuous at $y = y_0$. From (2.6) we have

$$\widetilde{p} = \psi U' - (U - c)\psi' \qquad \text{is continuous everywhere.} \tag{2.10}$$

If U is continuous we must have ψ continuous at $y = y_0$, but if U is discontinuous we must derive the kinematic condition as we did before. Perturbing the vortex sheet to $y = y_0 + \varepsilon h_0 e^{ik(x-ct)}$ we find on both sides of the sheet

$$\frac{h_0}{ik} = \frac{\psi}{U - c} \qquad \Longrightarrow \qquad \frac{\psi}{U - c} \ \text{ must be continuous everywhere.} \tag{2.11}$$

Necessary conditions for instability

As discussed earlier, we can determine whether the flow is unstable by searching *real* k for a corresponding wave speed c with $c_i > 0$.

Rayleigh's inflection-point theorem: *If the flow is unstable then* $U(y)$ *has an inflection point.*

Proof: Since the flow is unstable, we have for some real k with $c_i > 0$,

$$\psi'' - k^2\psi - \frac{U''}{U - c}\psi = 0 \ . \tag{2.12}$$

Multiplying by the complex conjugate ψ^* and integrating gives

$$\int_{y_1}^{y_2} \left(|\psi'|^2 + k^2|\psi|^2 + \frac{U''}{U - c}|\psi|^2\right) dy = 0 \tag{2.13}$$

after an integration by parts for the first term. Taking imaginary parts,

$$\int_{y_1}^{y_2} \frac{U'' \, \Im m(U - c^*)}{|U - c|^2} |\psi|^2 \, dy = 0 \quad \Longrightarrow \quad c_i \int_{y_1}^{y_2} U'' \left|\frac{\psi}{U - c}\right|^2 dy = 0. \tag{2.14}$$

Hence U'' must change sign (at least once).

Fjørtoft's theorem: If the flow is unstable then $U''(U - U_s) < 0$ for some value of y in (y_1, y_2), where y_s is a point at which $U''(y_s) = 0$, and $U_s = U(y_s)$.

Proof: The real part of (2.13) gives

$$\int_{y_1}^{y_2} \left(|\psi'|^2 + k^2|\psi|^2 + U''(U - c_r) \left| \frac{\psi}{U - c} \right|^2 \right) dy = 0. \tag{2.15}$$

Now $\int_{y_1}^{y_2} U''|\psi/(U-c)|^2 \, dy = 0$, so adding $(c_r - U_s)$ times this gives

$$\int_{y_1}^{y_2} \left(|\psi'|^2 + k^2|\psi|^2 + U''(U - U_s) \left| \frac{\psi}{U - c} \right|^2 \right) dy = 0 \tag{2.16}$$

from which the result follows.

Examples:
(i) Here $U'' < 0$ everywhere, so the flow is necessarily stable by Rayleigh's theorem.
(ii) Here $U'' = 0$ at y_s; but $U''(U - U_s) \geqslant 0$ everywhere so the flow is stable by Fjørtoft's theorem.
(iii) Here $U'' = 0$ at y_s, but $U''(U - U_s) \leqslant 0$ everywhere. This flow *might* be unstable.

Note that both theorems give only *necessary*, not sufficient, conditions for instability. $U = \sin y$ can be shown to satisfy the conditions but to be stable if $y_2 - y_1 < \pi$.

The Viscous Case: The Orr-Sommerfeld equation.

For a viscous fluid subject to a basic flow which is either plane Couette flow ($U = y$) or plane Poiseuille flow ($U = 1 - y^2$), an analysis similar to that above leads to the **Orr–Sommerfeld equation**

$$(U - c)(\psi'' - k^2\psi) - U''\psi = \frac{1}{ikR_e}\left(\psi'''' - 2k^2\psi'' + k^4\psi\right) \tag{2.17}$$

where $R_e = VL/\nu$ is the Reynolds number. It leads to a dispersion relation of the form $F(k,\, c;\; kR_e) = 0$ which depends on the value of R. We are interested in the values of k and R_e for which $c_i > 0$ giving instability. Usually we draw the **neutral stability curve**, where $c_i = 0$ in the parametric plane $(k,\, kR_e)$. Typically, there is a minimum value R_c of R_e above which some mode becomes unstable. The theory then predicts instability will set in for $R_e > R_c$.

The Orr-Sommerfeld equation can also be used to analyse the stability of boundary layers, for example to predict the behaviour of small disturbances in the Blasius layer on a semi-infinite flat plate. Although the flow in the boundary layer is not exactly parallel, it can be argued that correct results will be obtained by analysing the stability of a uni-directional flow which agrees with the $U(y)$-profile at some distance x along the layer. For the Blasius boundary layer, because of its self-similar structure and a layer thickness which increases as $x^{1/2}$, the critical Reynolds number can be interpreted as a critical distance along the plate at which instability will commence.

Squires' Theorem:

If a 3D-mode $e^{ikx+ilz}$ becomes unstable at a particular value of R_e, then there is an equivalent 2D-mode which is unstable at a smaller value of R_e. It therefore suffices to consider only 2-D modes.

Proof: Consider the 3-D perturbation

$$\mathbf{u} = (U(y),\, 0,\, 0) + \varepsilon(u(y),\, v(y),\, w(y))\, e^{ikx+ilz+st}\ . \tag{2.18}$$

Then the perturbed Navier-Stokes equations are

$$\left.\begin{aligned} iku + v' + ilw &= 0 \\ (s+ikU)u + vU' &= -ikp + R_e^{-1}(u'' - k^2u - l^2u) \\ (s+ikU)v &= \ -p' \ + R_e^{-1}(v'' - k^2v - l^2v) \\ (s+ikU)w &= -ilp + R_e^{-1}(w'' - k^2w - l^2w) \end{aligned}\right\} \tag{2.19}$$

We now define variables $\overline{u}$ and $\overline{p}$ such that

$$\overline{u} = (ku + lw)/\overline{k}, \qquad \overline{p} = \overline{k}p/k \qquad \text{where} \quad \overline{k}^2 = (k^2 + l^2)\ . \tag{2.20}$$

Then the first equation becomes $i\overline{k}\overline{u} + v' = 0$ while adding k times the 2nd equation to l times the 4th we obtain

$$(s+ikU)\overline{k}\overline{u} + kvU' = -i\overline{k}^2 p + R_e^{-1}\overline{k}(\overline{u}'' - \overline{k}^2\overline{u}) \tag{2.21}$$

or writing $\overline{R} = kR_e/\overline{k}$, and $\overline{s} = s\overline{k}/k$

$$\left.\begin{aligned} (\overline{s} + i\overline{k}U)\overline{u} + vU' &= -i\overline{k}\overline{p} + \overline{R}^{-1}(\overline{u}'' - \overline{k}^2\overline{u}) \\ (\overline{s} + i\overline{k}U)v &= -\overline{p}' + \overline{R}^{-1}(v'' - \overline{k}^2 v) \\ i\overline{k}\overline{u} + v' &= 0 \end{aligned}\right\} \tag{2.22}$$

Comparing (2.22) with (2.19), we see that the 3D problem to find $\overline{s}$ in terms of $\overline{k}$ and $\overline{R}$ is mathematically identical to the 2D problem obtained by setting $l = 0$ in (2.19). Furthermore, if we write $s = -ikc$, then $\overline{s} = -i\overline{k}c$.

Consider first the inviscid problem, setting $(R_e)^{-1} = 0$. Then if a mode (k, l) is unstable with growth rate $\Re e(s)$, then the mode $(\overline{k}, 0)$ must also be unstable with growth rate $\Re e(\overline{s})$. But as $\overline{k} \geqslant k$, **the 2D disturbance grows faster.**

If R_e is finite, then the 3-D mode (k, l) may go unstable at some critical Reynolds number $R_e = R_c$, corresponding to $\overline{R} = \overline{R}_c \equiv kR_c/\overline{k}$. Then by the mathematical similarity, the 2-D mode $(\overline{k}, 0)$ will go unstable at $R_e = \overline{R}_c$. But as $\overline{R}_c \leqslant R_c$, **an equivalent 2-D mode will go unstable at a lower Reynolds number.** We deduce that 2-D modes are the first to go unstable as R_e increases, and it is sufficient to consider these only.