M1A1: Solutions to Problem Sheet 6, Rotating Systems

i1. The outward centrifugal force is $mR\Omega^2$, which must be balanced by the normal reaction from the wall, N. The maximum frictional force which can act is then $\mu m R\Omega^2$, which must be greater than the person's weight. Thus

$$\mu m R\Omega^2 \geqslant mg \qquad \text{or} \qquad \Omega \geqslant \frac{g}{R\mu^2} \simeq 1.8$$

The cylinder must rotate about once every 3.5 seconds $(2\pi/\Omega)$. Note that the mass of the person is irrelevant.

2. The effective gravity is a resultant of $R\omega^2$ in the $\widehat{R}$-direction, and g vertically downwards. The component of this in a direction inclined at ψ to the horizontal is $R\omega^2\cos\psi - g\sin\psi$. This is zero if

$$\tan\psi = \frac{\omega^2 R}{g} \qquad \text{and the gradient of the surface is} \quad \tan\psi = f'(R)\ .$$

Integrating, we find the surface shape is $z = f(R) = z_0 + \frac{1}{2}\omega^2 R^2/g$, as required.

Consider now a bubble floating on the surface. As the surface is normal to the local gravity, there is no obvious reason why the bubbles should fall to the middle. Probably, it is an effect of the bubble protruding from the surface, and having its rotation slowed down by air resistance.

3. At a point on the equator, let z point towards the North, x towards the East, and then y points towards the centre of the earth. Neglecting terms quadratic in ω, the position vector $\mathbf{r}$ of the stone (measured in the rotating frame of the earth) obeys

$$\ddot{\mathbf{r}} = (\ddot{x},\, \ddot{y},\, \ddot{z}) = (0,\, g,\, 0) - 2(0,\, 0,\, \omega)\wedge(\dot{x},\, \dot{y},\, \dot{z})$$

(Or we could use the vector solution given in lectures.) Thus

$$\ddot{x} = +2\omega\dot{y} \qquad \ddot{y} = g - 2\omega\dot{x} \qquad \ddot{z} = 0$$

At $t = 0$, we have $(x, y, z) = (0, -h, 0)$ and $(\dot{x}, \dot{y}, \dot{z}) = (0,\, 0,\, 0)$.

Integrating, we have $z = 0$ for all time, $\dot{x} = 2\omega(y + h)$ and hence $\ddot{y} = g + O(\omega^2)$, so that $y = \frac{1}{2}gt^2 - h$. Thus $\dot{x} = \omega g t^2$ and $x = \frac{1}{3}\omega g t^3$. The stone hits the ground when $y = 0$ and hence $t = \sqrt{2h/g}$. At this time $x = \frac{1}{3}\omega\sqrt{8h^3/g}$ as required.

If $h = 50$, $g = 9.81$ and $\omega = 2\pi/(3600 \times 24)$, then $d \simeq 0.0077$ which is less than 1cm. Could save your life, though.

4. We have $\mathbf{v}_{rot} = \dot{r}\,\widehat{\mathbf{r}}$, and $\mathbf{a}_{rot} = \ddot{r}\,\widehat{\mathbf{r}}$, $\mathbf{r} = r\,\widehat{\mathbf{r}}$ and $\boldsymbol{\omega} = \dot{\theta}\widehat{\mathbf{z}}$. Furthermore, $\widehat{\mathbf{z}}\wedge\widehat{\mathbf{r}} = \widehat{\theta}$, and $\widehat{\mathbf{z}}\wedge\widehat{\theta} = -\widehat{\mathbf{r}}$. Thus

$$\begin{aligned}
\mathbf{a}_{in} &= \mathbf{a}_{rot} + \dot{\boldsymbol{\omega}}\wedge\mathbf{r} + 2\boldsymbol{\omega}\wedge\mathbf{v}_{rot} + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}) \\
&= \ddot{r}\,\widehat{\mathbf{r}} + \ddot{\theta}r\widehat{\theta} + 2\dot{\theta}r\widehat{\theta} + \dot{\theta}\widehat{\mathbf{z}}\wedge(\dot{\theta}r\widehat{\theta}) \\
&= (\ddot{r} - r\dot{\theta}^2)\,\widehat{\mathbf{r}} + (r\ddot{\theta} + 2r\dot{\theta}^2)\,\widehat{\theta} = (\ddot{r} - r\dot{\theta}^2)\,\widehat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\,\widehat{\theta}\ .
\end{aligned}$$