M1M1 Handout 3: Proof of Taylor's Theorem

We first prove that if a function $f(x)$ is $(n+1)$-times differentiable, and all these derivatives are continuous in some interval [a,b], then for x in this interval

$$f(x) = f(a) + (x-a)f'(a) + \ldots + \frac{1}{n!}(x-a)^n f^{(n)}(a) + \int_a^x \frac{f^{(n+1)}(t)(x-t)^n}{n!}\,dt. \quad (3.1)$$

We do this by induction. We note that when $n = 0$ the RHS is

$$f(a)+\int_a^x f'(t)\,dt = f(a)+\Big[f(t)\Big]_a^x = f(a)+f(x)-f(a) = f(x) \qquad \text{and equals the LHS.}$$

We next assume that (3.1) holds when $n = k$. Then integrating by parts, we have

$$\begin{aligned} f(x) &= f(a) + (x-a)f'(a) + \ldots + \frac{1}{k!}(x-a)^k f^{(k)}(a) + \int_a^x \frac{f^{(k+1)}(t)(x-t)^k}{k!}\,dt \\ &= f(a) + \ldots + \left[\frac{f^{(k+1)}(t)(x-t)^{k+1}}{k!(k+1)(-1)}\right]_a^x + \int_a^x \frac{f^{(k+2)}(t)(x-t)^{k+1}}{(k+1)!}\,dt \\ &= f(a) + \ldots + \frac{f^{(k+1)}(a)(x-a)^{k+1}}{(k+1)!} + \int_a^x \frac{f^{(k+2)}(t)(x-t)^{k+1}}{(k+1)!}\,dt \end{aligned}$$

which is (3.1) with n replaced by $k+1$. This completes the inductive step, and hence, by induction, (3.1) holds for all n. We now observe that since $f^{(n+1)}$ is a continuous function on the closed interval $[a,x]$ it attains its maximum value, say M, and its minimum value, say m, somewhere in this interval. Furthermore, it takes every value between m and M somewhere in this interval. (This intuitively obvious result, which we haven't proved, is called the intermediate value theorem, not to be confused with the mean value theorem!) Using standard properties of inequalities and integrals, we have if $x \geqslant t \geqslant a$,

$$\begin{aligned} & m \leqslant f^{(n+1)}(t) \leqslant M \qquad \Longrightarrow \qquad \frac{m(x-t)^n}{n!} \leqslant \frac{f^{(n+1)}(t)(x-t)^n}{n!} \leqslant \frac{M(x-t)^n}{n!} \\ \Longrightarrow \quad & \int_a^x \frac{m(x-t)^n}{n!}\,dt \leqslant \int_a^x \frac{f^{(n+1)}(t)(x-t)^n}{n!}\,dt \leqslant \int_a^x \frac{M(x-t)^n}{n!}\,dt \\ \Longrightarrow \quad & \frac{m(x-a)^{n+1}}{(n+1)!} \leqslant \int_a^x \frac{f^{(n+1)}(t)(x-t)^n}{n!}\,dt \leqslant \frac{M(x-a)^{n+1}}{(n+1)!}. \end{aligned}$$

Thus we have

$$m \leqslant \phi \leqslant M \qquad \text{where} \quad \phi = \frac{(n+1)!}{(x-a)^{n+1}} \int_a^x \frac{f^{(n+1)}(t)(x-t)^n}{n!}\,dt \quad \Longrightarrow \quad \phi = f^{(n+1)}(\xi)$$

for some value ξ since $f^{(n+1)}$ takes every value between m and M. Putting all this together, we have proved **Taylor's theorem:** If f is $(n+1)$-times differentiable, then for some ξ with $a < \xi < x$

$$f(x) = f(a) + (x-a)f'(a) + \ldots + \frac{1}{n!}(x-a)^n f^{(n)}(a) + \frac{(x-a)^{n+1} f^{(n+1)}(\xi)}{(n+1)!}. \quad (3.2)$$

The same result holds if $x < a$ with a ξ in $x < \xi < a$. Note: although we haven't yet formally defined integration in the course we will do so shortly.