M2M2 Notes 3: Adjoint Operators and Eigen-functions

Adjoint operators in general: Suppose u and v are elements of a space S. Let L be a linear operator from S to S, so that $L[\alpha u + \beta v] = \alpha L[u] + \beta L[v]$, for α and β constants. We also need an **inner product** mapping $S \otimes S$ to the real numbers $\Re$, which we write as $\{u, v\}$, with $\{u, u\} > 0$ unless $u = 0$. Then M is defined to be the **adjoint** of L if

$$\{u, L[v]\} \; = \; \{M[u], v\} \qquad \forall\, u,\, v \qquad \text{in} \quad S\ .$$

It is easy to see that the adjoint of $L_1 L_2$ is $M_2 M_1$ (note the reverse order; two operators do not necessarily commute, $L_1 L_2 \neq L_2 L_1$).

If $M = L$, then the operator L is said to be **self-adjoint** or **Hermitian**. Self-adjoint operators have several nice properties. For example, they have real eigenvalues, a fact which is important in Quantum Mechanics.

Example: Adjoints of matrices: Suppose $\mathbf{u}$ and $\mathbf{v}$ are real m–dimensional vectors with $\{\mathbf{u}, \mathbf{v}\} \; = \mathbf{u} \cdot \mathbf{v} \equiv \mathbf{u}^T \mathbf{v}$, where ^{T} denotes the transpose. Let L be a real $m \times m$ matrix. Then

$$\forall\, \mathbf{u}\,, \mathbf{v} \qquad \{\mathbf{u}, L\mathbf{v}\} = \mathbf{u}^T(L\mathbf{v}) = (L^T\mathbf{u})^T\mathbf{v} \quad \text{and so} \quad M = L^T\ .$$

If L is a symmetric matrix, then $L = L^T = M$ and L is self–adjoint.

Now suppose we allow the elements of L, $\mathbf{u}$ and $\mathbf{v}$ to be complex. We must then be careful because if $\mathbf{u} = (1, i, 0)$, say, then $\mathbf{u} \cdot \mathbf{u} = 1^2 + i^2 = 0$, even though $\mathbf{u} \neq 0$. Instead we define an inner product $\{\mathbf{u}, \mathbf{v}\} = \overline{\mathbf{u}} \cdot \mathbf{v} \equiv \overline{\mathbf{u}}^T \mathbf{v}$, where $^-$ denotes the complex conjugate. [Note $\{u, v\}$ is now a complex number, although $\{u, u\} \geqslant 0$ is real.] We now have

$$\{\mathbf{u}, L\mathbf{v}\} = \overline{\mathbf{u}}^T(L\mathbf{v}) = (L^T\overline{\mathbf{u}})^T\mathbf{v} = \overline{(\overline{L^T}\mathbf{u})^T}\mathbf{v} \quad \text{and so} \quad M = \overline{L^T}\ .$$

Thus L is self-adjoint if $L = \overline{L^T}$. An example is $L = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$.

Adjoints of differential operators: Now let u and v be (possibly complex) functions of x on the interval $(a,\, b)$ and let L be a linear differential operator. An appropriate **inner product** is

$$\{u, v\} \; = \int_a^b \overline{u} v \, dx \qquad \text{so that} \quad \{u, u\} = \int_a^b |u|^2 dx \geqslant 0\ . \tag{6.4}$$

We can find the adjoint to L using integration by parts. For example, suppose that $L[\phi] = f(x) d\phi/dx$. Then

$$\{u, L[v]\} \; = \int_a^b \overline{u} f \frac{dv}{dx}\, dx = \Big[\overline{u} f v\Big]_a^b - \int_a^b v \frac{d}{dx}(f\overline{u}) dx = \Big[\overline{u} f v\Big]_a^b - \int_a^b v\, \overline{\left(\frac{d}{dx}(\overline{f} u)\right)}\, dx\ .$$

If we ignore the boundary term for the moment, then we can identify

$$M[\phi] \equiv -\frac{d}{dx}(\overline{f}\,\phi) = -\overline{f}\,\frac{d\phi}{dx} - \frac{d\overline{f}}{dx}\,\phi\ .$$

The operator M is called the **adjoint** of L, or more precisely, the **formal adjoint** of L. The term adjoint should strictly be used only if boundary conditions are associated with each operator such that the terms evaluated at a and b vanish, so that $\{u, Lv\} = \{Mu, v\}$. Thus suppose $L \equiv f d/dx$ is associated with the boundary condition $v(a) = 0$. The boundary term $\left[\overline{u} f v\right]_a^b = 0$ if we require that u satisfies $u(b) = 0$. We can then say that for $a < x < b$, the operator

$$L[\phi] \equiv f(x)\frac{d\phi}{dx} \quad \text{with} \quad \phi(a) = 0 \quad \text{has the adjoint} \quad M[\phi] \equiv -\frac{d(\overline{f}\phi)}{dx} \quad \text{with} \quad \phi(b) = 0 \ .$$

Now $M = L$ only if f is an imaginary constant, in which case we say L is **formally self-adjoint.** The term self-adjoint should only be used if the same boundary conditions apply. For example, the operator $L[\phi] \equiv i d\phi/dx$ with the **periodic boundary condition** $\phi(a) = \phi(b)$ is **self-adjoint,** as then $\overline{u}(a) i v(a) = \overline{u}(b) i v(b)$.

The general second order linear differential operator can be written as

$$L[\phi] \equiv a_0(x)\phi'' + a_1(x)\phi' + a_2(x)\phi \ . \tag{6.5}$$

We obtain its adjoint by integrating by parts twice, so that $\{u, Lv\} =$

$$\int_a^b \overline{u}(a_0 v'' + a_1 v' + a_2 v)dx = \Big[\overline{u} a_0 v' + \overline{u} a_1 v\Big]_a^b + \int_a^b \Big[a_2 \overline{u} v - v'(a_0\overline{u})' - v(a_1\overline{u})'\Big] dx$$

$$= \Big[\overline{u} a_0 v' + \overline{u} a_1 v - v(a_0\overline{u})'\Big]_a^b + \int_a^b v\left[(a_0\overline{u})'' - (a_1\overline{u})' + a_2\overline{u}\right] dx \ .$$

We restrict attention to the case when a_0, a_1 and a_2 are real. The **formal adjoint** to L is

$$M[\phi] = (a_0\phi)'' - (a_1\phi)' + a_2\phi = a_0\phi'' + (2a_0' - a_1)\phi' + (a_2 - a_1' + a_0'')\phi \ .$$

Comparing with (6.5), we see that L is **formally self-adjoint** if $a_0 = a_0$, $2a_0' - a_1 = a_1$ and $a_2 - a_1' + a_0'' = a_2$. All of these are satisfied if $a_1 = a_0'$. Replacing a_0 by p, a_2 by q and L by S, the general **formally self-adjoint** real second order differential operator can be written

$$S[\phi] = \frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi \ . \tag{6.6}$$

The boundary conditions for which S is self-adjoint on (a, b) must be such that

$$\Big[p\overline{u}v' - pv(\overline{u})'\Big]_a^b = 0 \ . \tag{6.7}$$

Differential eigenvalue problems: We are interested in the problem $Ly + \lambda y = 0$, where L is given by (6.5). It is more convenient to deal with self-adjoint operators. It is possible to find a positive function $w(x)$ such that $wL[y] \equiv S[y]$, is self-adjoint (see the problem sheet). The resulting eigen-value equation, called the **Sturm-Liouville Equation**, is

$$S[y] + \lambda w y \equiv \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y + \lambda w(x)y = 0 \qquad \text{for} \quad a < x < b \ . \tag{6.8}$$

It is associated with homogeneous boundary conditions such that (6.7) holds whenever u and v satisfy them. We require the functions $p(x) > 0$ and $w(x) > 0$ for $a < x < b$. The eigen-values, λ_n, and eigen-functions, y_n, of (6.8) are of considerable importance.