M2M2 Notes 1: Series Solutions of Bessel's Equation

Bessel functions have wide application in applied mathematics, especially when dealing with cylindrical geometry. They are solutions to **Bessel's equation**, which is

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0 \qquad \text{for} \quad \nu \geqslant 0 \; . \tag{3.10}$$

In our standard notation, $p(x) = 1/x$ and $q(x) = 1 - \nu^2/x^2$, so that $x = 0$ is a **regular singular point**. Using the test we derived (3.3), we see that the **point at infinity** is an **irregular singular point**, because $x^2 q(x)$ is infinite as $x \to \infty$.

We seek a series solution about $x = 0$ of the form

$$y(x) = x^\lambda \sum_{r=-\infty}^{\infty} a_r x^r \qquad \text{with} \quad a_0 \neq 0 \qquad \text{and} \quad a_r = 0 \text{ for } r < 0 \; . \tag{3.11}$$

Substituting this in (3.10), we obtain

$$\sum_{-\infty}^{\infty}(\lambda + r)(\lambda + r - 1)a_r x^{\lambda+r} + \sum_{-\infty}^{\infty}(\lambda + r)a_r x^{\lambda+r} + \sum_{-\infty}^{\infty} a_r x^{\lambda+r+2} - \nu^2 \sum_{-\infty}^{\infty} a_r x^{\lambda+r} = 0 \; ,$$

or equating coefficients of $x^{\lambda+s}$,

$$[(\lambda + s)^2 - \nu^2]a_s + a_{s-2} = 0 \; . \tag{3.12}$$

When $s < 0$, we obtain $0 = 0$. When $s = 0$, as $a_0 \neq 0$ we must have

$$\lambda^2 - \nu^2 = 0 \; ,$$

the **indicial equation**, giving two roots, $\lambda = \pm\nu$. The difference between these two roots, $\lambda_1 - \lambda_2 = 2\nu$, and so we should be prepared for expansions like (3.11) to give only one valid solution if 2ν is an integer. Putting $s = 1$ in (3.12), we get $[(\lambda + 1)^2 - \nu^2]a_1 = 0$ and so $a_1 = 0$, as $2\lambda + 1 \neq 0$ if 2ν is not an integer. For $s \geqslant 2$, we have, using $\lambda^2 = \nu^2$

$$s(s \pm 2\nu)a_s = -a_{s-2} \; . \tag{3.13}$$

Since $a_1 = 0$, this tells us $a_s = 0$ whenever s is odd. Two independent solutions of (3.10) when 2ν is not an integer are therefore

$$\left.\begin{aligned} y_1(x) &= x^\nu \left[1 - \frac{x^2}{2(2 + 2\nu)} + \frac{x^4}{2(4)(2 + 2\nu)(4 + 2\nu)} + \ldots\right] \\ y_2(x) &= x^{-\nu} \left[1 - \frac{x^2}{2(2 - 2\nu)} + \frac{x^4}{2(4)(2 - 2\nu)(4 - 2\nu)} + \ldots\right] \end{aligned}\right\} . \tag{3.14}$$

These solutions are proportional to the **Bessel functions of the first kind** $J_\nu(x)$ and $J_{-\nu}(x)$, defined by

$$J_\nu(x) = \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{1}{2}x)^{\nu+2r}}{r!(\nu + r)!} = \frac{1}{\nu!}\left(\frac{x}{2}\right)^\nu \left[1 - \frac{(\frac{1}{2}x)^2}{\nu + 1} + \frac{(\frac{1}{2}x)^4}{2(\nu + 1)(\nu + 2)} + \ldots\right] \; , \tag{3.15}$$

and similarly for $-\nu$. Note that $\nu!$ is defined for non-integral ν by the **Gamma Function**,

$$\nu! \equiv \Gamma(\nu+1) \equiv \int_0^\infty t^\nu e^{-t} dt \qquad \text{for } \nu \text{ not a negative integer.}$$

The series in (3.14) and (3.15) converge for all x, as $\lim_{r\to\infty} |a_r/a_{r-2}| = 0$, from (3.12).

Order Integer + Half

Now consider $\nu = N + \frac{1}{2}$, for some integer $N \geqslant 0$, so that 2ν is an integer. The resulting functions are sometimes called 'Spherical Bessel Functions' as they can arise when using spherical polar coordinates. Surprisingly, the expansions (3.14) still give two independent solutions. This is because the series only involve even powers of x and the possible "resonance" does not occur. Interestingly, these solutions can be written in terms of trigonometric functions. For example, when $\nu = \frac{1}{2}$, $y_1 = \sin x/x^{1/2}$ and $y_2 = \cos x/x^{1/2}$. [See question 3 on the first problem sheet.]

Integer Order

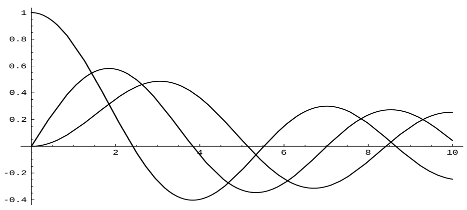
When $\nu = N$, a positive integer, however, the expansion for y_2 in (3.14) is invalid, and the logarithms appear. The function $J_N(x)$ remains analytic, and is very important. For example, the Bessel function of order 0 has the expansion

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r(\frac{1}{2}x)^{2r}}{(r!)^2} \; . \tag{3.16}$$

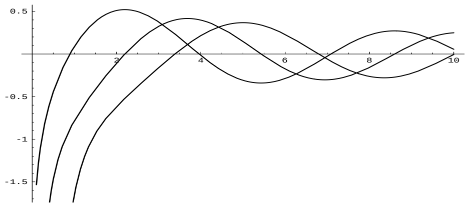
This is the equal roots case, and a second solution, $Y_0(x)$, is usually written

$$Y_0(x) = \frac{2}{\pi}\left(J_0(x)[\ln(\tfrac{1}{2}x) + \gamma] + \sum_{r=1}^{\infty} b_r x^{2r} \right) \; ,$$

for suitable constants b_r, which we could calculate. The constants $2/\pi$ and γ are arbitrary, and need not concern us. Similar expansions hold for $Y_N(x)$ in terms of the Bessel function J_N. The solution Y_N is called the **Bessel function of the second kind**. It is singular at the singular point $x = 0$, $Y_N \sim x^{-N}$ for $N > 0$, whereas J_N is analytic.



J_0, J_1 and J_2



Y_0, Y_1 and Y_2

As $x \to \infty$, both J_N and Y_n behave like $x^{-1/2}(A \sin x + B \cos x)$. Note that between any two zeros of J_N there is precisely one zero of Y_N and vice versa (see problem sheet 1 question 2).