M2M2 Notes 2: Convergence of Fourier Series; Gibbs' Phenomenon

Warning: This sheet contains Analysis, which some readers may find offensive.

We consider the partial sums of the Fourier series, $S_N(x)$, defined for $0 < x < 2\pi$ by

$$S_N(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx) \quad \text{where} \quad \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \frac{1}{\pi}\int_0^{2\pi} f(t) \begin{pmatrix} \cos nt \\ \sin nt \end{pmatrix} dt \ .$$

We are interested in the limit of S_N as $N \to \infty$, which we would like to be close to $f(x)$. Substituting for the coefficients and inverting the order of the sum and the integral (no problem for a **finite** sum), we have, using $\cos nx \cos nt + \sin nx \sin nt \equiv \cos n(t-x)$,

$$S_N(x) = \frac{1}{\pi}\int_0^{2\pi} f(t) D_N(t-x)\, dt \qquad \text{where} \quad D_N(\theta) = \tfrac{1}{2} + \sum_{n=1}^{N} \cos n\theta \ ,$$

Now $2 \sin \frac{1}{2}\theta \, D_N(\theta) = \sin(N+\frac{1}{2})\theta$ (see Problem sheet 3). We can therefore write

$$S_N(x) = \frac{1}{2\pi}\int_0^{2\pi} f(t) \frac{\sin[(N+\frac{1}{2})(t-x)]}{\sin[\frac{1}{2}(t-x)]}\, dt = \frac{1}{2\pi}\int_{-x}^{2\pi - x} f(x+s) \frac{\sin[(N+\frac{1}{2})s]}{\sin \frac{1}{2}s}\, ds \ , \quad (4.11)$$

substituting $s = t - x$. Note that the integrand of (4.11) is finite even at $s = 0$. To understand the behaviour of this integral when N is large, we need to generalise (4.6):

Riemann's Lemma: (RL) If $g(s)$ is an integrable function in the interval $[a, b]$, then

$$\lim_{\alpha \to \infty} \int_a^b g(s) \sin(\alpha s + \beta)\, ds = 0 \ . \qquad (4.12)$$

We sketch the proof of (4.12). First, we note that if $g(s) = g_1$, a constant, then

$$\left| \int_a^b g_1 \sin(\alpha s + \beta)\, dx \right| = \left| \frac{g_1}{\alpha}[\cos(\alpha a + \beta) - \cos(\alpha b + \beta)] \right| \leqslant \frac{2|g_1|}{\alpha} \ . \qquad (4.13)$$

Next, we recall that for a function to be integrable, it can be approximated to arbitrary accuracy by functions $g_M(x)$ which are constant over each of M intervals which partition (a, b). On each interval over which g_M is constant we can use (4.13). Now

$$\left| \int_a^b g(s) \sin(\alpha s + \beta)\, ds \right| \leqslant \int_a^b |g - g_M| dx + \left| \int_a^b g_M(s) \sin(\alpha s + \beta)\, ds \right| \ .$$

The first integral can be made small for large M, and then the second $\to 0$ as $\alpha \to \infty$. **(RL)** follows. It can be extended to include infinite integrals as, say, $b \to \infty$.

We now write $\alpha = N + \frac{1}{2}$, $\beta = 0$, and, for fixed x, $g(s) = f(x+s)/\sin \frac{1}{2}s$ in (4.11). We can't use **(RL)** directly on (4.11) as $g(s)$ is not integrable over a range including $s = 0$, since $g(s) \sim 1/s$ as $s \to 0$. We therefore exclude the point $s = 0$ by an arbitrarily small interval $(-\delta,\, \delta)$ for $1 \gg \delta > 0$ and split the integral in (4.11) into four ranges as follows: $\int_{-x}^{2\pi - x} = \int_{-x}^{-\delta} + \int_{-\delta}^{0} + \int_{0}^{\delta} + \int_{\delta}^{2\pi - x}$. Consider first $\int_{\delta}^{2\pi - x} g(s) \sin \alpha s \, ds$. The function $g(s)$ is integrable in this range, and so, by **(RL)**, this integral tends to zero as $\alpha \to \infty$. Similarly, $\int_{-x}^{-\delta} \to 0$. The behaviour of $S_N(x)$ for large values of N is thus determined by the behaviour of $f(x)$ in the arbitrarily small interval $(x - \delta, x + \delta)$. We now consider

$$I \equiv \frac{1}{2\pi}\int_0^\delta f(x+s)\frac{\sin \alpha s}{\sin \frac{1}{2}s}ds = \frac{1}{\pi}\int_0^\delta f(x+s)\frac{\sin \alpha s}{s}ds + \frac{1}{2\pi}\int_0^\delta f(x+s)\sin \alpha s\left[\frac{1}{\sin\frac{1}{2}s} - \frac{2}{s}\right]ds$$

The last integral tends to zero by **(RL)** as the problem at $s = 0$ has been removed. For $\varepsilon > 0$, we denote by $f(x+)$ the limit as $\varepsilon \to 0$ of $f(x + \varepsilon)$. Then

$$\frac{1}{\pi}\int_0^\delta f(x+s)\frac{\sin \alpha s}{s}ds = \frac{1}{\pi}\int_0^\delta \left[\frac{f(x+s) - f(x+)}{s}\right]\sin \alpha s\, ds + \frac{f(x+)}{\pi}\int_0^\delta \frac{\sin \alpha s}{s}\, ds\ .$$

The first of these terms will tend to zero as $\alpha \to \infty$ by **(RL)**, provided $[f(x+s) - f(x+)]/s$ is integrable. This will certainly be the case if f has a right-derivative at x. If so

$$\lim_{\alpha\to\infty} I = \lim_{\alpha\to\infty}\frac{f(x+)}{\pi}\int_0^\delta \frac{\sin \alpha s}{s}\, ds = \frac{f(x+)}{\pi}\int_0^\infty \frac{\sin u}{u}du = \tfrac{1}{2}f(x+)\ , \tag{4.14}$$

substituting $u = \alpha s$ and using contour integration (see Problem Sheet 3). Similarly, $\int_{-\delta}^{0} \to \frac{1}{2}f(x-)$. We have therefore proved a **Fourier Convergence Theorem**:

If f is an integrable function for which both right and left derivatives exist at x, then its Fourier series converges to $\frac{1}{2}[f(x+) + f(x-)]$. Note:

(1) If f is continuous at x then the series converges to $f(x)$.
(2) When $x = 0$ or $x = 2\pi$ the series coverges to $\frac{1}{2}[f(0+) + f(2\pi-)]$.
(3) We obtain this result if we adopt the dubious interpretation "$\frac{1}{2} + \sum_{n=1}^{\infty} \cos n\theta = \pi\delta(\theta)$". Note that this series converges for **no** values of θ. This is not analysis, but may be useful.

The Gibbs Phenomenon: We now consider the function $f(x) = 1$ in $0 < x < \pi$ and $f(x) = -1$ in $\pi < x < 2\pi$, whose Fourier series we calculated in (4.9). This function has discontinuities at $x = k\pi$ for integers k, near which we saw from the computer simulation the series behaves strangely, overshooting the required jump by the same amount for all N. From (4.9) and problem sheet 3, when x is small and positive and N is large, we have

$$S_{(2N-1)}(x) = \frac{2}{\pi}\int_0^x \frac{\sin 2Ns}{\sin s}\, ds \simeq \frac{2}{\pi}\int_0^{2Nx}\frac{\sin \xi}{\xi}\, d\xi\ . \tag{4.15}$$

If we fix x and let $N \to \infty$, $S_{(2N-1)} \to 1$, as it should. However, if we fix N we can then ask for which value of x is $S_{(2N-1)}$ a maximum? The integrand changes sign at $\xi = \pi$, and so $S_{(2N-1)}(x)$ has a maximum when $2Nx = \pi$. For this value of x, $S_{(2N-1)} \simeq 1.179$, a value independent of N. We therefore have

$$\lim_{N\to\infty} S_N(x) = f(x) \qquad \forall x, \qquad \text{but} \quad \lim_{N\to\infty}\left[\max_x[S_N(x)]\right] \simeq 1.179\ \max_x[f(x)]\ .$$

Counter-intuitive results such as this show us that analysis is important, although sometimes difficult.