M2M2 Notes 5: Separable solutions in Cylindrical and Spherical Polars.

We have seen that the wave equation $u_{tt} = c^2\nabla^2 u$ for $u(\mathbf{x}, t)$, reduces to the Helmholz equation, $\nabla^2 u + k^2 u = 0$, if we seek solutions (often called **normal modes of vibration**) with $u \propto \exp(i\omega t)$, where $k = \omega/c$. We now seek separable solutions to this equation in spherical and cylindrical polar coordinate systems. Laplace's equation, for which $k = 0$, is a special case, while the diffusion equation, $u_t = \nabla^2 u$ with $u \propto \exp(-k^2 t)$ is similar.

In terms of **cylindrical polar coordinates** (r, θ, z), the Helmholz equation is

$$\nabla^2 u + k^2 u \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0\ . \tag{7.15}$$

We look for a separable solution $u(r, \theta, z) = R(r)\Theta(\theta)Z(z)$. Substituting in (7.15) and dividing through by $R\,\Theta\,Z$, we obtain

$$\frac{R''}{R} + \frac{R'}{rR} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} + k^2 = 0\ .$$

Now the only terms which depend on z and θ are the fourth and third respectively. Both Z''/Z and Θ''/Θ must therefore be constant. We require 2π-periodicity in the θ-direction so that $\Theta'' + n^2\Theta = 0$, where $n \geqslant 0$ is an integer, and $\Theta = C_n\cos(n\theta) + D_n\sin(n\theta)$. In the z-direction we may have either exponential or trigonometric behaviour. We write $Z'' - \lambda^2 Z = 0$, so that $Z = E\exp(\lambda z) + F\exp(-\lambda z)$, if λ is real, or if λ is imaginary, $\lambda = i\nu$, then we write $Z = E\cos\nu z + F\sin\nu z$. If $\lambda = 0$, we have $Z = E + Fz$.

Substituting for R and Z in (7.15) and rearranging, we obtain the equation for $R(r)$

$$r^2R'' + rR' + \mu^2r^2R - n^2R = 0 \qquad \text{where} \quad \mu^2 = \lambda^2 + k^2 = k^2 - \nu^2\ . \tag{7.16}$$

This we recognise as Bessel's equation of order n with the argument μr, so that the independent solutions are $J_n(\mu r)$ and $Y_n(\mu r)$, the latter being singular at $r = 0$. The general solution of (7.15) can be written as a sum of separable solutions,

$$u = \sum_{n=0}^{\infty}\sum_{\lambda,\nu}\left[AJ_n(\mu r) + BY_n(\mu r)\right]\left[C\cos n\theta + D\sin n\theta\right]\left\{\begin{array}{c} Ee^{\lambda z} + Fe^{-\lambda z} \\ E\cos\nu z + F\sin\nu z\end{array}\right\} \tag{7.17}$$

where the coefficients A, B, C, D, E, F may vary with n and λ. Now if we take the sines in the z-direction, then μ is real only if $k^2 - \nu^2 \geqslant 0$, which may not always be the case. In particular, if we are solving the Laplace equation ($k = 0$) then the Bessel functions in (7.17) will have imaginary arguments. We must then introduce **modified Bessel functions** $I_n(x)$ and $K_n(x)$, which are solutions of $x^2y'' + xy' - (n^2 + x^2)y = 0$. Loosely speaking I_n and K_n behave respectively like $\exp(x)$ and $\exp(-x)$ for large x, and K_n is singular at $x = 0$ but I_n is regular. We can then write down the general solution to Laplace's equation in cylindrical polar coordinates,

$$u = \sum_{n=0}^{\infty}\sum_{\lambda,\nu}\left\{\begin{array}{c} AJ_n(\lambda r) + BY_n(\lambda r) \\ AI_n(\nu r) + BK_n(\nu r)\end{array}\right\}\Big[C\cos n\theta + D\sin n\theta\Big]\left\{\begin{array}{c} Ee^{\lambda z} + Fe^{-\lambda z} \\ E\cos\nu z + F\sin\nu z\end{array}\right\} \tag{7.18}$$

We now consider **spherical polar coordinates**, $(r,\ \theta,\ \phi)$. The symbols r and θ denote different quantities from in cylindrical polars, but ϕ is essentially the same as the cylindrical polar angle θ. In these coordinates, the Helmholz equation, $\nabla^2 u + k^2 u = 0$, for $u(r,\ \theta,\ \phi)$ is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} + k^2 u = 0\ . \tag{7.19}$$

We try a separable solution $u = R(r)\Theta(\theta)\Phi(\phi)$ in (7.19) and multiply by r^2/u to find

$$\frac{(r^2 R')'}{R} + \frac{(\sin\theta\,\Theta')'}{\sin\theta\,\Theta} + \frac{1}{\sin^2\theta}\frac{\Phi''}{\Phi} + k^2 r^2 = 0\ .$$

Now ϕ appears only as Φ''/Φ which must equal $-m^2$, where m, for 2π-periodicity is an integer, $m \geqslant 0$ (m corresponds to n overleaf). Separating the terms which depend on r from those which depend on θ, we must have, for some λ,

$$(r^2 R')' + k^2 r^2 R = \lambda R \qquad \text{and} \qquad \frac{(\sin\theta\,\Theta')'}{\sin\theta} + \left(\lambda - \frac{m^2}{\sin^2\theta}\right)\Theta = 0\ . \tag{7.20}$$

We consider first the equation for Θ. If we define $x = \cos\theta$ and $y(x) = \Theta(\theta)$, we obtain $[(1-x^2)y']' + [\lambda - m^2/(1-x^2)]y = 0$ which is known as the **Associated Legendre equation** whose solutions are **Associated Legendre functions**. When $m = 0$, a regular solution is $y = P_n(x)$, where P_n is a Legendre polynomial, provided $\lambda = n(n+1)$ for integral n. It turns out that for (7.20) to have solutions which are regular at both $x = \pm 1$ (or $\theta = 0,\ \pi$), we must still require $\lambda = n(n+1)$ and also $n \geqslant m \geqslant 0$. The regular solution is written $P_n^m(x)$, while a second solution, which is singular at $x = \pm 1$, is written $Q_n^m(x)$. It can be shown that

$$P_n^m(\cos\theta) = (-\sin\theta)^m \frac{d^m}{dx^m}P_n(x) \qquad \text{where} \quad x = \cos\theta\ .$$

We now consider the equation for $R(r)$. When $k = 0$, so that we are dealing with Laplace's equation, (7.20) is homogeneous in r and thus has solutions $R = r^p$ where $p(p+1) = n(n+1)$ so that either $p = n$ or $p = -(n+1)$. The solutions to Laplace's equation are thus

$$u = \sum_{n=0}^{\infty}\sum_{m=0}^{n}[A_{mn}r^n + B_{mn}r^{-(n+1)}][C_{mn}P_n^m + D_{mn}Q_n^m][E_{mn}\cos m\phi + F_{mn}\sin m\phi]\ . \tag{7.21}$$

When $k \neq 0$, $R(r)$ satisfies $r^2R'' + 2rR' + [k^2r^2 - n(n+1)]R = 0$, which apart from the factor of 2 resembles Bessel's equation. The solutions are called **spherical Bessel functions**, and are written $j_n(kr)$ and $y_n(kr)$. They are related to Bessel functions of half-integer order, $j_n(x) = (\pi/2x)^{1/2}J_{n+1/2}(x)$, and can be expressed in terms of elementary functions, e.g. $j_0(x) = (\sin x)/x$ and $y_0 = (\cos x)/x$. The general solution of (7.19) is thus

$$u = \sum_{n=0}^{\infty}\sum_{m=0}^{n}[Aj_n(kr) + By_n(kr)][CP_n^m + DQ_n^m][E\cos m\phi + F\sin m\phi]\ . \tag{7.22}$$

The regular **axi-symmetric** solutions for which $m = 0$ and $D = 0$ in (7.21) and (7.22) are particularly important. The functions $P_n^m(\cos\theta)\exp(\pm im\phi)$ are often called **spherical harmonics.**