1. The differential operator L is defined by

$$Ly \equiv x^2 y'' - xy' + y \qquad \text{in} \quad 0 < x < \infty \ .$$

Show that $y = x$ is a solution to the homogeneous equation $Ly = 0$, and use the method of variation of parameters to find y such that

$$Ly = x \qquad \text{with} \quad y(1) = 1 \ , \qquad \text{and} \quad y'(1) = 1 \ .$$

2. Find a series solution about $x = 0$ for the equation

$$xy'' + (1-x)y' + \lambda y = 0 \qquad \text{in} \quad x > 0 \qquad \text{with} \quad y(0) = 1 \ ,$$

where λ is constant.

What is the radius of convergence of your series?

Determine the eigenvalues, λ, for which the series terminates as a polynomial.

Rewrite the equation in Sturm-Liouville form, and deduce the orthogonality relation between the eigenfunctions.

3. By summing a complex geometric series or otherwise, show that

$$\sum_{n=1}^{N} \sin n\theta = \frac{\cos \frac{1}{2}\theta - \cos\left[(N+\frac{1}{2})\theta\right]}{2\sin\frac{1}{2}\theta} \ .$$

Integrating between x and π and letting $N \to \infty$, infer that

$$-\ln\left|2\sin\tfrac{1}{2}x\right| = \sum_{n=1}^{\infty} \frac{\cos nx}{n} \ , \qquad (*)$$

stating carefully any assumptions about limits that you make.

[**NB**: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots = \ln 2$.]

Sketch over several periods the 2π-periodic function whose Fourier series is given by $(*)$.

By considering one of the Fourier coefficients, deduce that

$$\int_0^{\pi} \ln[\sin t]\, dt = -\pi \ln 2 \ .$$

4. The function $f(x)$ is defined for positive constants a and b by

$$f(x) = \begin{cases} e^{-ax}\sin bx & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x < 0 \,. \end{cases}$$

Show that its Fourier transform is

$$F(k) \equiv \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx = \frac{b}{(ik+a)^2 + b^2} \,.$$

The function $y(x)$ tends to zero as $x \to \pm\infty$ and obeys the equation

$$y'' + 2ay' + q^2 y = g(x) \,,$$

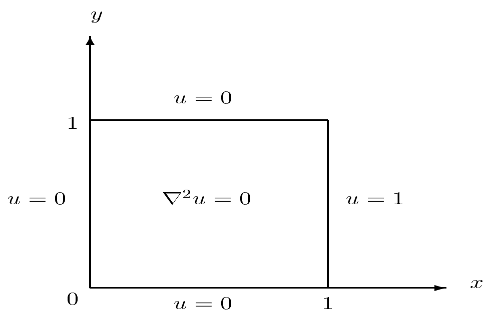
where a and q are constants with $q > a > 0$. Show that

$$y(x) = \frac{1}{b}\int_{-\infty}^{\infty} f(x-s) g(s)\, ds \qquad \text{where} \quad b = \sqrt{q^2 - a^2}$$

and f is as given above.

5. The function $u(x,\, y)$ is zero on three sides of the square in the figure, while $u = 1$ on $x = 1$. Inside the square, it satisfies Laplace's equation,

$$\nabla^2 u \equiv u_{xx} + u_{yy} = 0 \ .$$



Find $u(x,\, y)$, and show that

$$u\left(\tfrac{1}{2},\, \tfrac{1}{2}\right) = \frac{4}{\pi} \sum_{\substack{n=1 \\ n \ odd}}^{\infty} \frac{\sin \frac{1}{2} n\pi}{n} \, \frac{\sinh[\frac{1}{2} n\pi]}{\sinh[n\pi]} \ .$$

By combining this problem with three very similar problems, deduce that

$$\tfrac{1}{16}\pi = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \, \frac{\sinh[(k+\frac{1}{2})\pi]}{\sinh[(2k+1)\pi]} \ .$$