1. The function $y(x)$ satisfies the equation

$$Ly \equiv (x+2)y'' + (x+1)y' - y = -(x+2)^2 e^{-x}$$

and the boundary conditions

$$y'(0) = y(0) \qquad \text{and} \quad y \to 0 \qquad \text{as} \quad x \to \infty \; .$$

Show that the homogeneous equation $LY = 0$ has a solution $Y = e^{-x}$, and hence find $y(x)$.

2. Show that $x = 0$ is a Regular Singular Point of the equation

$$4xy'' + 2y' + y = 0 \; .$$

Find two independent series solutions of this equation in the form

$$y = x^{\lambda} \sum_{n=0}^{\infty} a_n x^n \; .$$

Simplify the solutions using the power series for $\cos t$ and $\sin t$.

3. Find the Fourier series for the 2π-periodic function

$$f(x) = |x| - \tfrac{1}{2}\pi \qquad \text{in} \qquad -\pi < x < \pi\ .$$

Using the series you have found, derive the Fourier series of

$$g(x) = \begin{cases} +1 & \text{for} \quad \pi > x > 0 \\ -1 & \text{for} \quad -\pi < x < 0 \end{cases} .$$

Describe qualitatively the behaviour of partial sums of this second series near $x = 0$.

State Parseval's identity, and use it to show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^4}{96}\ .$$

4. The functions $f(x)$ and $g(x)$ have Fourier transforms

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \qquad \text{and} \quad G(k) = \int_{-\infty}^{\infty} g(y) e^{-iky} dy \ .$$

Define $h(x)$, the **convolution** of $f(x)$ and $g(x)$, and show that

$$F(k)G(k) = \int_{-\infty}^{\infty} h(s) e^{-iks} ds \ .$$

Calculate $F(k)$ and $G(k)$ when f and g are given by

$$f(x) = \left\{ \begin{array}{ccc} e^{-x} & \text{for} & x > 0 \\ 0 & \text{for} & x < 0 \end{array} \right\} \qquad \text{and} \quad g(x) = x f(x) \ .$$

For these functions f and g, find the function $p(x)$ which satisfies the equation

$$\int_{0}^{\infty} f(y) p(x-y) dy = g(x) \ .$$

5. Find the eigen-values, λ_n, and corresponding eigen-functions $y_n(x)$, for the problem

$$y'' + 2\alpha y' + \lambda y = 0 \qquad \text{in} \quad 0 < x < 1 \ ,$$

where α is a given constant, subject to the boundary conditions

$$y(0) = y(1) = 0 \ .$$

Write the equation in Sturm-Liouville form, and hence state the orthogonality relation between y_n and y_m.

Consider now the PDE for $u(x,\, t)$,

$$u_t = u_{xx} + 2\alpha u_x \qquad \text{in} \quad t > 0, \quad 0 < x < 1$$

with boundary conditions

$$u(0,\, t) = u(1,\, t) = 0 \qquad \text{and} \quad u(x, 0) = f(x) \ .$$

Obtain the solution as an infinite sum of separable solutions, giving the coefficients in terms of $f(x)$.