1. One solution of Legendre's equation of order 1,

$$L[y] \equiv (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0 \ ,$$

is the Legendre polynomial $y = P_1(x) \equiv x$. Use the method of variation of parameters to find the solution to

$$L[y] = x \qquad \text{in} \quad 0 < x < 1$$

subject to the boundary conditions $\dfrac{dy}{dx} = 0$ at $x = 0$, and $y(1)$ is finite.

Verify that in fact $y(1) = \frac{1}{3}(1 - \ln 2)$.

Identify another solution, $y = Q_1(x)$, to the homogeneous equation $L[y] = 0$ which is logarithmically singular at $x = \pm 1$.

2. For what values of λ does the equation

$$xy'' - (2x + b)y' + 2y = 0 \ ,$$

where b is a constant, have a power series solution of the form

$$y = x^{\lambda} \sum_{n=0}^{\infty} a_n x^n \qquad \text{with} \quad a_0 \neq 0 \ ?$$

In each case, find a recurrence relation for the coefficients and calculate the first few terms of the series.

For what values of b are there two independent solutions of this form?

Find the general solution when $b = 1$ in terms of power series, and relate it to the function e^{2x}.

3. Find the Fourier series of the 2π–periodic function defined in $(-\pi,\,\pi)$ by

$$f(x) = \begin{cases} \dfrac{1}{2h} & \text{for} \quad |x| < h \\ 0 & \text{for} \quad h < |x| < \pi \end{cases}$$

where h is a constant such that $0 < h < \pi$.

By considering the value of this series when $x = \frac{1}{2}\pi$, determine the sum of the series

$$S(h) \equiv \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin(2kh) \ ,$$

taking care to consider all possible values of h in the range $0 < h < \pi$.

Discuss the behaviour of $f(x)$ and its Fourier series as $h \to 0$.

4. Find the Fourier transform $F(k)$ of the function

$$f(x) = \begin{cases} 1 & \text{for} \quad |x| < a \\ 0 & \text{for} \quad |x| > a \end{cases}$$

where a is constant.

What do the value of $F(0)$ and the behaviour of $F(k)$ as $k \to \infty$ indicate about the function $f(x)$?

The function $u(x,\, t)$ satisfies the partial differential equation

$$u_{tt} = u_{xx} \qquad \text{in} \quad t > 0, \quad -\infty < x < \infty$$

and the initial conditions

$$u(x,\, 0) = 0 \ , \qquad u_t(x,\, 0) = g(x) \ .$$

Defining the Fourier transform with respect to x,

$$\widehat{u}(k,\, t) = \int_{-\infty}^{\infty} e^{-ikx}\, u(x,\, t)\, dx \ ,$$

show that

$$\widehat{u}(k,\, t) = \frac{\sin kt}{k}\, G(k) \ ,$$

where $G(k)$ is the transform of $g(x)$.

Use the convolution theorem to find $u(x,\, t)$.

5. A square drum occupies the region $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant a$. When struck, its normal displacement $u(x,\, y,\, t)$ obeys the wave equation

$$u_{tt} = c^2(u_{xx} + u_{yy}) \ , \qquad \text{where } c \text{ is constant,}$$

while $u = 0$ on the square boundary. Find the separable solutions to this problem, and hence the possible frequencies ω of oscillations proportional to $\exp(i\omega t)$.

At time $t = 0$, the drum is struck so that

$$u(x,\, y,\, 0) = 0 \ , \qquad u_t(x,\, y,\, 0) = f(x,\, y) \ .$$

Find an expansion of $u(x,\, y,\, t)$ in terms of separable solutions, expressing the coefficients as double integrals involving $f(x,\, y)$.

If the drum is struck along the diagonal $x = y$, so that the initial velocity is a delta-function, $f(x,\, y) = \delta(x - y)$, show that

$$u(x,\, y, t) = \sum_{n=1}^{\infty} \frac{\sqrt{2}}{\pi c n} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi\sqrt{2}ct}{a}\right) \ .$$

$$\left[\text{Note that} \quad \int_0^a \int_0^a p(x)q(y)\delta(x-y)\, dx\, dy = \int_0^a p(x)q(x)\, dx \ .\right]$$

Explain very briefly why striking the drum in this particular manner is likely to lead to a more musical sound.