1. If the Sturm-Liouville system,

$$y'' + \frac{\lambda y}{1-x^2} = 0 \qquad \text{in} \quad -1 < x < 1 \qquad \text{with} \quad y(\pm 1) = 0 \ ,$$

has non-zero solutions $y = y_m(x)$ for $\lambda = \lambda_m$ and $y = y_n(x)$ for $\lambda = \lambda_n$, write down the orthogonality relation between y_m and y_n.

If $y_m = 1 - x^2$, find the value of λ_m. Use the method of variation of parameters to find all solutions to

$$y'' + \frac{2y}{1-x^2} = 5x^2 - 1 \qquad \text{with} \quad y(\pm 1) = 0 \ .$$

Show that one of these solutions is itself an eigen-function, $y_n(x)$ for which $\lambda_n = 12$.

Verify that the orthogonality relation between y_m and y_n holds.

2. Find a series solution about $x = 0$ for the problem

$$(1 - x^2)y'' + \lambda y = 0 \ , \qquad \text{where } \lambda \text{ is a given constant.}$$

Find the values of λ for which the series terminates in a polynomial of degree n, $y = y_n(x)$.

Writing $x = 1/z$ and $Y(z) = y(x)$, obtain an ODE satisfied by $Y(z)$. What kind of point is $z = 0$?

Find the values of λ for which there exists a solution of the form

$$Y(z) = \sum_{m=p}^{\infty} a_m z^m \ , \qquad \text{where } p \geqslant 0 \text{ is an integer.}$$

3. Find the Fourier series of the function $y(x) = 1 - x^2$ for $|x| < 1$.

Write down Parseval's theorem for this interval, and infer that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \ .$$

By integrating the series you have found, show that

$$x^3 - x = \frac{12}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi x)}{n^3} \qquad \text{in} \quad |x| < 1 \ .$$

Using Parseval's theorem once more, deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} \ .$$

4. Find the Fourier transforms of $f''(x)$ and $x^2 f(x)$ in terms of the transform of $f(x)$ defined by

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx \ .$$

Assuming that solutions valid for all x can be found for the ODE

$$(1 - x^2) y''(x) + \lambda y = 0 \qquad \text{with } \lambda \text{ constant,}$$

find a differential equation satisfied by the transform $Y(k)$ of $y(x)$.

Writing $U(k) = k^{3/2} Y$, show that U satisfies the Bessel equation of order ν,

$$U'' + \frac{U'}{k} + U\left[1 - \frac{\nu^2}{k^2}\right] = 0 \qquad \text{where} \quad \nu^2 = \lambda + \tfrac{1}{4} \ .$$

State those values of λ for which $U(k)$ is expressible in terms of elementary functions.

Use the inversion formula to write down an integral representation of a solution $y(x)$ in terms of the Bessel function $J_\nu(k)$.

5. Spheroidal coordinates c and t, which you have not met before, are defined for $-1 \leqslant c \leqslant 1$ and $t \geqslant 1$, with surfaces of constant t describing spheroids (egg-shapes).

A function $\phi(c,\, t)$ inside the spheroid $t = t_0$ obeys the equation

$$(1-c^2)\frac{\partial^2 \phi}{\partial c^2} - (1-t^2)\frac{\partial^2 \phi}{\partial t^2} = 0 \qquad \text{for} \quad 1 < t < t_0, \quad -1 < c < 1$$

with boundary conditions

$$\phi(\pm 1, t) = 0, \quad \phi(c, 1) = 0, \qquad \text{and} \quad \phi(c, t_0) = f(c) \ ,$$

where $f(c)$ is a given function. Find the separable solutions to this problem satisfying the homogeneous boundary conditions in terms of the eigenfunctions $y_n(x)$ of the problem

$$(1-x^2)y'' + \lambda y = 0 \qquad \text{with} \quad y(\pm 1) = 0 \ .$$

Hence find $\phi(c,\, t)$ as an infinite sum, expressing the coefficients in terms of integrals of $f(c)$, and evaluate it when $f(c) = 1 - c^2$.