1. Show, by substitution, that x^2 and $(1-x)$ are solutions of the equation

$$x(2-x)\frac{d^2y}{dx^2} - 2(1-x)\frac{dy}{dx} - 2y = 0 \ .$$

Show that the Wronskian of these solutions is non-zero in the range $0 < x < 2$.

By the method of variation of parameters, find the general solution of the equation

$$x(2-x)\frac{d^2y}{dx^2} - 2(1-x)\frac{dy}{dx} - 2y = x^2 - 4x \ .$$

Deduce that the solution satisfying the boundary conditions

$$y(0) = 0 \qquad \text{and} \quad y(2) = 0$$

is

$$y = x^2 \ln\left(\frac{2}{|x|}\right) \ .$$

2. Find two independent solutions $y_1(x)$ and $y_2(x)$ of the equation

$$y'' + y\left(1 - \frac{2}{x^2}\right) = 0 \ ,$$

as power series about $x = 0$. Obtain the recurrence relations between the coefficients and give three terms of each series.

What is the radius of convergence of these series?

Verify that

$$y_3 = \cos x - \frac{\sin x}{x}$$

is a solution to the ODE, and express y_3 in terms of y_1 and y_2.

3. Using integration by parts, or otherwise, show that

$$\int_{-\pi}^{\pi} \cosh x \cos(nx)\, dx = (-1)^n \frac{2\sinh\pi}{1+n^2} \, .$$

Hence determine the Fourier series of the function

$$f(x) = \cosh x \qquad \text{in} \qquad -\pi < x < \pi \, .$$

What is the significance of the behaviour of the Fourier coefficients as $n \to \infty$?

Obtain two representations of $\sinh x$ by differentiating and by integrating the series term by term. By comparing the two (and NOT otherwise), infer the Fourier series for $g(x) = x$, in $-\pi < x < \pi$.

4. Define the Fourier transform, $F(k)$, of a function $f(x)$ and derive formulae for the transforms of $f'(x)$ and $xf(x)$ in terms of $F(k)$.

Deduce similar formulae for the transforms of $f''(x)$ and $xf'(x)$.

The function $y(x)$ satisfies the equation

$$4y'' + (2x-2)y' + (2-x)y = 0 \ ,$$

together with the integral condition

$$\int_{-\infty}^{\infty} y(x)\, dx = 1 \ .$$

Obtain an equation satisfied by $Y(k)$, the Fourier transform of $y(x)$, and show that

$$Y(k) = \exp\left(-k^2\right) \ .$$

Hence find $y(x)$.

$$\left[\text{Note: For any real } \alpha, \quad \int_{-\infty+i\alpha}^{\infty+i\alpha} \exp(-t^2)\, dt = \sqrt{\pi}\right] \ .$$

5. When the male of a certain species of "mole cricket" wishes to attract a mate, he digs an underground chamber of suitable acoustic properties and bangs his wings together. Assuming this chamber is a sphere of radius a centred on the cricket, the sound produced is determined by the solution $p(r,t)$ to the equation

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \nabla^2 p = \frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r}$$

with the boundary conditions

$$p = 0 \qquad \text{on} \quad r = a, \qquad p \ \text{ finite at } r = 0.$$

The sound speed, c, is a known constant. Seek separable solutions of the form $p(r,t) = R(r)T(t)$ to this equation and show that the frequencies, ω_n, of the sound produced by the cricket are determined by the eigenvalues of a Sturm-Liouville problem for $R(r)$.

Write this problem in standard form, and deduce an orthogonality relation satisfied by two eigenfunctions, $R_n(r)$ and $R_m(r)$.

Find the eigenvalues and eigenfunctions explicitly using the substitution $R(r) = u(r)/r$.

Discuss briefly whether or not the sound produced will appear musical to human ears.