1. Prove that the Wronskian of any two solutions of the equation

$$y'' + q(x)y = 0 \qquad \text{is constant.}$$

Suppose $y = y_1(x)$ is one solution of this equation, with $y_1(0) \neq 0$. Show that the solution to the problem

$$y'' + q(x)y = f(x) \qquad \text{with} \quad y(0) = y'(0) = 0$$

can be written in the form

$$y = y_1(x) \int_0^x \left(\frac{1}{y_1^2(s)} \int_0^s f(t) y_1(t)\, dt \right) ds \; .$$

Simplify this expression when

(a) $f(x) = y_1'(x)$

and

(b) $f(x) = q(x)$.

2. Using a power series about the point $x = 0$, show that there is a unique solution of the equation

$$xy'' + (2 - x)y' + \lambda y = 0 \qquad \text{in} \quad 0 < x < \infty$$

satisfying $y(0) = 1$, where λ is a given constant.

Find the radius of convergence of this series, and give the terms up to and including x^3 explicitly.

Show that there are certain eigenvalues, $\lambda = \lambda_n$, for which the series terminates as a polynomial, $y = y_n(x)$. By writing the equation in self-adjoint form, determine the orthogonality relation between two eigenfunctions.

3. Sketch the functions $x \sin x$ and $x \cos x$ for $-\pi \leqslant x \leqslant \pi$.

Calculate the Fourier series of the function $x \sin x$ in $-\pi \leqslant x \leqslant \pi$.

By differentiation, deduce that the Fourier series for $x \cos x$ over this range is

$$x \cos x = -\tfrac{1}{2} \sin x + \sum_{n=2}^{\infty} \frac{2n(-1)^{n+1}}{1-n^2} \sin nx \ .$$

By considering $\int_{-\pi}^{\pi} x^2 (\cos^2 x + \sin^2 x)\, dx$ and using Parseval's theorem, show that

$$\sum_{n=2}^{\infty} \frac{1+n^2}{(1-n^2)^2} = \tfrac{1}{6}\pi^2 - \tfrac{5}{8} \ .$$

[You may recall that $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$.]

4. State the relations between the Fourier transform of $f(x)$ and those of $f'(x)$ and $xf(x)$.

Find the transform of the function $g(x) = e^{-|x|}$.

The function $y = K_0(x)$ is an even function of x and satisfies

$$xy'' + y' - xy = 0 \qquad \text{with} \quad \int_{-\infty}^{\infty} K_0(x)\, dx = \pi \ .$$

Show that $Y(k)$, the transform of $y(x)$, is given by

$$Y(k) = \frac{\pi}{(1+k^2)^{1/2}} \ .$$

Use the convolution theorem to evaluate

$$h(x) = \int_{-\infty}^{\infty} K_0(s) K_0(x-s)\, ds \ ,$$

and deduce that

$$\int_0^{\infty} \left[K_0(s)\right]^2 ds = \tfrac{1}{4}\pi^2 \ .$$

5. The surface of a plutonium sphere, of radius a, is kept at constant temperature. Heat is generated internally by radioactive decay at a rate proportional to the temperature and diffuses with coefficient D. The temperature distribution $u(r,\, t)$ therefore obeys

$$\frac{\partial u}{\partial t} = Qu + \frac{D}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) \qquad \text{in} \quad 0 < r < a \ ,$$

where D and Q are known constants, and satisfies the conditions

$$u(a,\, t) = 0 \ , \qquad u(0,\, t) \quad \text{finite} \qquad \text{and} \quad u(r,\, 0) = f(r) \ ,$$

for some given function $f(r)$.

Obtain the solution as a sum of separable solutions, expressing the coefficients in terms of $f(r)$.

[You may find it useful to substitute $u(r,\, t) = \frac{1}{r}v(r)T(t)$.]

Determine the condition for a catastrophic meltdown, for which the plutonium temperature grows without limit as $t \to \infty$.