1. The function $y(x)$ obeys the inhomogeneous differential equation

$$Ly \equiv x(1-x)y'' - (1-x)y' - y = x^2 \qquad \text{in} \quad 0 < x < 1 \ . \qquad (1)$$

Find constants a and b such that $y = ax + b$ solves the homogeneous equation $Ly = 0$. Hence find the general solution to (1).

Deduce that the particular solution satisfying the boundary conditions $y = 0$ at $x = 0$ and at $x = 1$ is

$$y = x(1-x) + (1-x)\ln(1-x) \ .$$

2. The modified Bessel function $I_0(x)$ is the solution to the equation

$$xy'' + y' - xy = 0 \tag{2}$$

satisfying $y(0) = 1$. Express I_0 as a power series about $x = 0$, giving the general term explicitly. Discuss the region of convergence of the series.

Write down the form of $K_0(x)$, a solution of (2) independent of I_0. You need not calculate its coefficients.

By differentiating (2), show that $I_0'(x)$ and $K_0'(x)$ are solutions to the equation

$$x^2y'' + xy' - (1 + x^2)y = 0 \ . \tag{3}$$

Deduce that a solution to (3) which is finite at $x = 0$ is

$$y = \sum_{n=1}^{\infty} \frac{n(x/2)^{2n-1}}{(n!)^2} \ .$$

3. Find the Fourier series of the 2π-periodic function

$$f(x) = x \qquad \text{in} \qquad -\pi < x < \pi \; .$$

Hence solve the diffusion equation for $u(x,\, t)$

$$u_t = u_{xx} \qquad \text{in} \quad t > 0 \; , \qquad 0 < x < \pi$$

subject to the boundary and initial conditions

$$u(0,t) = u(\pi,t) = 0 \qquad \text{and} \quad u(x,0) = x \; .$$

4. Defining the Fourier transform of the function $f(x)$ by

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx \ ,$$

find the function whose transform is

$$F(k) = e^{-\alpha|k|} \qquad \text{where} \quad \alpha > 0 \ .$$

What feature of $F(k)$ is reflected by the behaviour of $f(x)$ as $x \to \infty$?

State the convolution theorem and hence find $u(x)$ given that

$$\frac{1}{x^2+2} = \int_{-\infty}^{\infty} \frac{u(s)}{(x-s)^2+1}\, ds \ .$$

5. The Sturm-Liouville equation

$$\big(p(x)y'\big)' + q(x)y + \lambda w(x)y = 0 \qquad \text{in} \quad a < x < b$$

has eigenvalues $\lambda = \lambda_n$ and eigenfunctions $y = y_n(x)$. Show that

$$\lambda_n = I[y_n] \qquad \text{where} \quad I[y] = \frac{\int_a^b \big(p(y')^2 - qy^2\big)\,dx}{\int_a^b wy^2\,dx} \;,$$

for boundary conditions on y to be specified.

Assuming that the eigenvalues are stationary values of $I[y]$, and that $I[y]$ is bounded below, explain briefly how to obtain approximations to the smallest eigenvalue λ_1.

Using a suitable quadratic trial function, show that the smallest eigenvalue of the problem

$$y'' + \lambda y = 0 \qquad \text{in} \quad 0 < x < \pi \qquad \text{with} \quad y'(0) = 0 \qquad \text{and} \quad y(\pi) = 0$$

satisfies

$$\lambda_1 \leqslant \frac{5}{2\pi^2} \;.$$

Find the exact eigenvalues and eigenfunctions for this problem, and comment on the accuracy of the approximation $\lambda_1 \simeq 5/(2\pi^2)$.