M2M2 Notes 5: If music be the food of mathematics...

Sound is a collection of vibrations, or pressure waves. A fluctuation $p(\mathbf{x}, t)$ to the ambient pressure obeys the wave equation,

$$c^2\nabla^2 p = p_{tt} \qquad \text{in some region } D, \quad \text{for } t > 0, \tag{7.4}$$

where c is the speed of sound, which we will assume to be constant. If we look for separable solutions to (7.4) of the form $p(\mathbf{x}, t) = u(\mathbf{x})T(t)$ then we find

$$\nabla^2 u + k^2 u = 0 \ , \qquad\qquad T'' + \omega^2 T = 0 \qquad \text{where} \quad \omega = ck \ .$$

The equation for u is known as the **Helmholtz equation**. It can be shown that the Laplacian is **self-adjoint** and that the eigenvalues (or **wave-numbers**) k^2 are positive. The time dependence is therefore $T = A\cos\omega t + B\sin\omega t$ and the solutions are oscillatory (wave-like) with a frequency ω determined by the possible eigenvalues k^2.

The eigenvalues k^2 depend on the domain D and the boundary conditions to be applied on the boundary. The general solution to (7.4) consists of a superposition of all the possible waves with all the possible frequencies. Whether or not the sound that is preduced is 'musical' or not depends on how these frequencies relate to each other.

Most musical instruments are essentially one-dimensional, depending on the vibration of a string or a thin column of air. We begin by considering the vibration of a string of length L (see also M2A1). Let $u(x, t)$ obey

$$u_{tt} = c^2 u_{xx} \qquad \text{in} \quad 0 < x < L, \ t > 0 \tag{7.5}$$

with the boundary conditions

$$u(0, t) = 0 \qquad \text{and} \quad u(L, t) = 0 \ , \tag{7.6}$$

and the initial conditions

$$u(x, 0) = f(x) \qquad \text{and} \quad u_t(x, 0) = g(x) \ . \tag{7.7}$$

Here u is the normal displacement of the string, which is held fixed at $x = 0$ and $x = L$. The initial configuration of the string is $u = f$, while its initial normal velocity is $u_t = g(x)$.

Looking for separable solutions of (7.5) of the form $u(x, t) = X(x)T(t)$, we find

$$\frac{T''}{T} = \frac{c^2 X''}{X} = C, \qquad \text{which must be constant,}$$

as the LHS is a function only of t while the RHS is formally a function of x. The function $X(x)$ is therefore either exponential or trigonometric. As we want to impose $X = 0$ at both $x = 0, L$, in the usual way we must have $0 > C = -k^2$. Then

$$X(x) = \alpha\cos kx + \beta\sin kx \qquad \text{and} \quad X(0) = 0 \quad \Longrightarrow \quad \alpha = 0 \ .$$

Then imposing $X(L) = 0$ gives $\beta \sin kL = 0$, and for non-trivial solutions we must have $k = n\pi/L$ for $n = 1, 2, \ldots$ The equation for $T(t)$ is similar. When we use the linearity to sum over all possible sperable solutions we therefore have

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right) , \qquad (7.8)$$

where A_n and B_n are constants to be found. Applying the initial conditions (7.7), we have

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \qquad \text{and} \quad g(x) = \sum_{n=1}^{\infty} B_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right) ,$$

so that A_n and B_n are essentially the Fourier coefficients of f and g. If we are interested in the motion resulting from a particular f and g, we calculate A_n and B_n in the usual way, when (7.8) gives the solution.

From a musical point of view, A_n and B_n give the amplitudes of a particular note (or frequency). The important feature is that no matter how the string is struck, or released from rest, the resulting sound is a mixture of notes whose frequencies are integer multiples (nc/L) of some lowest frequency, c/L, called the **fundamental**. Note that the boundary conditions (7.6) were vital in this regard. The same frequencies result if $u_x = 0$ at $x = 0, L$, (flute/recorder) but $u = 0$ at $x = 0$ and $u_x = 0$ at $x = L$ gives a different sound (clarinet/oboe).

The musical scale

I assume most of you are familiar with the concept of an octave. A note an octave above another note has **twice the frequency**. Middle C has a frequency of roughly 256Hz (cycles per second) so that all the Cs are roughly powers of 2. The **equitempered** musical scale is divided into twelve **semitones**, each of whose frequencies is a constant multiple, r, of the note a semitone lower. For twelve semitones to be equivalent to an octave, this requires $r^{12} = 2$, or $r = 2^{1/12}$.

Consider now the vibrating string, and suppose that its lowest frequency is C, for example. The next note generated when the string is plucked has twice the frequency, and is therefore another C, an octave higher. Likewise the fourth and eighth harmonics are Cs, but octaves higher. The notes corresponding to $n = 3$ and $n = 6$ are not C, but one is an octave higher than the other. Likewise $n = 5$ is a different note. If the vibrating string is to sound musical, it is important that the notes corresponding to $n = 3$ and $n = 5$ should 'harmonise' with the fundamental $n = 1$.

It is a curious fact that $2^{19} \simeq 3^{12}$. As a result, a note of 3 times the frequency is almost exactly 19 semitones higher. If the fundamental is a C, the third harmonic $n = 3$ is a G, an octave higher. Also $5 \simeq 2^{28/12}$. Thus the harmonic $n = 5$ is close to two octaves and 4 semitones higher than $n = 1$. This corresponds to an E. Thus the first 6 harmonics when a string vibrates correspond to C, E and G in various permutations. This you may recognise as the notes of C-major, the chord based on the fundamental. One-dimensional vibrations therefore sound well together.

In contrast, the note produced by a drum (a two-dimensional instrument) is much less pure, and not so easily pleasing to the ear. As we shall see, Bessel functions are to blame!