M2M2: Problem Sheet 1

Wronskians, Variation of Parameters and Green's Functions

1. If y_1 and y_2 are linearly independent solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0 \ , \tag{1}$$

with associated Wronskian $W(x)$, show that

$$y_2(x) = y_1(x)\left[\int^x \frac{W(t)}{y_1^2(t)}\,dt \qquad + \text{constant.}\ \right]$$

2. If y_1 and y_2 are linearly independent solutions of (1) above, and $y_2(\alpha) = 0$ and $y_2(\beta) = 0$ but $y_2(x) \neq 0$ in $\alpha < x < \beta$, prove that y_1 has precisely one zero in $\alpha < x < \beta$. [Hint: assume y_1 has no such zeros, use the last question and Rolle's theorem applied to (y_2/y_1) to derive a contradiction.
Next, assume y_1 has 2 such zeros, and derive a contradiction concerning y_2.]

3. Use a substitution of the form $y = \alpha(x)u$ to transform Bessel's equation of order n,

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \ ,$$

to an equation for $u(x)$ in normal form. Hence find the general solution in $x > 0$ of

$$x^2y'' + xy' + (x^2 - \tfrac{1}{4})y = 3x^{3/2}\sin x \ .$$

4. Show that a change of *independent* variable of the form $t = f(x)$ for a suitable function f can reduce (1) to the normal form

$$\frac{d^2y}{dt^2} + Q(t)y = 0 \ .$$

Hence find the general solution of the equation

$$y'' + y' - e^{-2x}y = 0 \ .$$

*5. Every solution of a particular Nth order, linear ODE can be written in the form

$$y = A\sin^{-1}x + B\cos^{-1}x + C\cos 2x + D\cos^2 x + E + F\left(\cos^4 x - \sin^4 x\right) + \int_G^x e^{-t^2}\,dt$$

and all values of the constants A, B, $C, \ldots G$ define such a solution. What is the value of N?
[A few lecturers got this question wrong when I showed it to them. Can you do better?]

6. The equation

$$Ly \equiv x^2(1-x)y'' + 2x(2-x)y' + 2(1+x)y = 0$$

has a solution of the form $y = x^m$ where m is constant. Find m, and hence the general solution of the inhomogenous equation $Ly = 1$.

7. Since the coefficients of the equation

$$(1-x)y'' + xy' - y = 2(1-x)^2 e^{-x} \qquad \text{in} \quad x > 1$$

sum to zero, it follows that $y = e^x$ is a solution of the homogeneous version. Deduce the solution satisfying the boundary condition $y \to 0$ as $x \to \infty$.

8. Find the boundary value Green's function $G(x,\, s)$ satisfying

$$G'' = \delta(x-s) \qquad \text{with} \quad G(0,s) = G(1,s) = 0$$

by requiring G to be continuous at $x = s$, while

$$\left[\frac{\partial G}{\partial x}\right]_{x=s} = 1\ .$$

Hence (and **not** otherwise) find the solution to the simple problem

$$y'' = 1 \qquad \text{in} \quad 0 < x < 1 \qquad \text{with} \quad y(0) = 0,\ y(1) = 0\ .$$