M2M2: Problem Sheet 2

Power Series Solutions; Method of Frobenius

1. Find the general solution, in the form of a power series about $x = 0$, of Legendre's equation of order n,

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0 \ .$$

State the condition on n for the equation to have a polynomial solution.

2. Find two linearly independent series solutions about $x = 0$ of the equation

$$xy'' + by' - y = 0 \qquad (1)$$

where the constant b is not an integer, giving the first four terms of each series.

3. Obtain two independent solutions of (1) when $b = 1$, including terms up to x^2 in the non-analytic solution.

***4.** **Either** by expanding the integrand as a power series in x and evaluating the coefficients of x^r **or** by demonstrating that the integral satisfies the requisite ODE, show that the Bessel function

$$J_0(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \sin u)\, du \ .$$

5. Seek a solution of the form $y = \sum_{k=0}^{\infty} a_k x^k$ to the equation

$$x^2 y'' - (1 - 3x)y' + y = 0 \ .$$

What is the radius of convergence of the series you obtain? What kind of point is $x = 0$?

Show that $x = \infty$ is a regular singular point of the equation, and find a solution as a series in $z = (1/x)$ which is analytic at $z = 0$.

***6** Seek a solution $y(x) = \exp(-\frac{1}{2}x^2)u(x)$ where $u(x) = \sum_{n=0}^{\infty} a_n x^n$ to the equation

$$y'' + (E - x^2)y = 0 \qquad \text{for some constant } E.$$

Show that the power series for u terminates if $E = 1, 3, 5, \ldots$ If this does not occur, show that $a_{2k+2} \sim a_{2k}/k$ for large k, and infer that $u(x)$ behaves like $\exp(x^2)$ for large x. Deduce that solutions such that $y \to 0$ as $|x| \to \infty$ exist only if E takes the above values.

[This problem is important in Quantum Mechanics. The values of E are the permitted energy levels of the harmonic oscillator.]