M2M2: Problem Sheet 3; Fourier Series

1. Find the Fourier series of the function $f(x)$ where

$$f(x) = 0 \qquad \text{for} \quad -\pi < x < 0 \qquad \text{and} \quad f(x) = 1 \qquad \text{for} \quad 0 < x < \pi \ .$$

Using Parseval's theorem, deduce that $\sum\limits_{n=1}^{\infty} (2n-1)^{-2} = \frac{1}{8}\pi^2$.

2. Show that if $-\pi < x < \pi$,

$$x^2 = \tfrac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \ ,$$

and by integrating and differentiating this series, infer the Fourier series of (a) x and (b) x^3 in the same interval.

3. Find the half-range sine series of $\cos x$. Comment on the behaviour of the coefficients as $n \to \infty$.

4. Using the complex form of the Fourier series, show that for $0 < x < 2\pi$,

$$e^x = \frac{1}{2\pi}\left(e^{2\pi} - 1\right) \sum_{-\infty}^{\infty} \frac{e^{inx}}{1 - in} \ .$$

By considering a particular value of x, deduce that

$$\pi \coth \pi \equiv \frac{\pi(e^{2\pi} + 1)}{e^{2\pi} - 1} = 1 + 2\sum_{n=1}^{\infty} \frac{1}{1+n^2} \ .$$

5. With respect to the printed sheet on convergence of Fourier series:
(a) Show that

$$\tfrac{1}{2} + \sum_{n=1}^{N} \cos n\theta = \frac{\sin[(N+\frac{1}{2})\theta]}{2\sin\frac{1}{2}\theta} \qquad \text{and} \qquad \sum_{n=1}^{N} \cos(2n-1)\theta = \frac{\sin 2N\theta}{2\sin\theta} \ .$$

*(b) Use contour integration to prove that (where $\Im m$ denotes the imaginary part)

$$\int_0^{\infty} \frac{\sin z}{z} dz = \tfrac{1}{2}\Im m \left[\int_{-\infty}^{\infty} \frac{e^{iz}}{z} dz\right] = \tfrac{1}{2}\pi \ .$$

6. Consider the $2L$-periodic square pulse function $f(x) = 1$ for $0 \leqslant |x| < 1$, $f(x) = 0$ for $1 < |x| < L$. Calculate the Fourier coefficients a_n. Defining $k_n = n\pi/L$, and $F_n = La_n$, show that as $L \to \infty$ the discrete points $(k_n,\ F_n)$ define the curve $F(k) = 2\sin k/k$.
[This illustrates the connection between Fourier Series and Fourier Transforms.]