M2M2: Problem Sheet 4; Fourier Transforms

1. Find the Fourier transform, $F(k) \equiv \int_{-\infty}^{\infty} \exp(-ikx) f(x)\, dx$, of the function

$$(\#) \qquad f(x) = \begin{cases} e^{-x} & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x < 0 \end{cases} .$$

Verify the Rayleigh-Parseval theorem, that $\int_{-\infty}^{\infty} |f|^2 dx = (1/2\pi) \int_{-\infty}^{\infty} |F|^2 dk$.
[Note that $F \sim k^{-1}$ as $k \to \infty$, because $f(x)$ is discontinuous.]

2. Show that the transform of the "triangle function,"

$$f(x) = \begin{cases} 1 - |x| & \text{for} \quad |x| < 1 \\ 0 & \text{for} \quad |x| > 1 \end{cases}$$

is

$$F(k) = \frac{2(1 - \cos k)}{k^2} .$$

Verify the relation between $F(0)$ and $\int_{-\infty}^{\infty} f(x)\, dx$.
[Here $F \sim k^{-2}$as $k \to \infty$, because $f'(x)$ is discontinuous.]

3. Deduce from the previous question the transform of $(1 - \cos x)/x^2$.

4. Suppose a function $f(x)$ has a Fourier transform $F(k)$. Show that the transform of $f(ax)$ is $F(k/a)/|a|$, where a is a real, non-zero constant.
[This illustrates how stretching of x corresponds to compression of k and vice versa.]

5. Similarly, show that $f(x - a)$ has the transform $\exp(-ika)F(k)$.
[This is often called the "shift theorem."]

6. If $f(x)$ is as given in (#) above, and $g(x) = xf(x)$, use the convolution theorem to find the function $p(x)$ which satisfies

$$\int_0^{\infty} f(y) p(x - y)\, dy = g(x) .$$

7. Find the **Laplace** transform of $\exp(i\alpha x)$, for constant α, and deduce the Laplace transforms of $\cos \alpha x$ and $\sin \alpha x$.

Use Laplace transforms to solve the problem

$$y'' + y = \sin \alpha x \qquad \text{in} \quad y > 0 \qquad \text{with} \quad y(0) = 1, \; y'(0) = 1 .$$