M2M2: Problem Sheet 5; Sturm-Liouville Equations

1. Consider the differential eigen-value problem

$$y'' + k^2 y = 0 \qquad \text{with} \quad y(0) = 0, \quad 2y'(0) = y(1) \ .$$

Demonstrate that the eigen-values satisfy the equation $2k = \sin k$ and show graphically that this equation has no real solutions other than $k = 0$. Does it have an imaginary solution, $k = i\alpha$? Why does this not conflict with the general Sturm-Liouville theory?

2. Show that $L[y] \equiv a_0 y'' + a_1 y' + a_2 y$ can be made formally self-adjoint by multiplying by an 'integrating factor' $w(x)$, such that $wL[y] = S[y] \equiv (py')' + qy$. Deduce that $w = (1/a_0)\exp(\int^x [a_1(t)/a_0(t)]dt)$, and use this to write the following equation in self-adjoint form: (k^2 is the eigen-value)

$$y'' - \frac{2}{x}y' + \left(\frac{2}{x^2} + k^2\right) y = 0 \qquad \text{with} \quad y'(0) = 0 \qquad \text{and} \quad y(\pi) = 0 \ .$$

Hence write down the orthogonality relation between two eigen-functions y_n and y_m.

3. Use the substitution $y(x) = xu(x)$ to solve the equation in question 3, and hence find the eigen-functions and eigen-values. Verify that the above orthogonality relation holds.

4. The functions $y_n(x)$ for $n = 0,\ 1,\ 2\ldots$ satisfy the equation

$$\left[(1 - x^2)y_n'\right]' + n(n+1)y_n = 0 \qquad \text{in} \quad |x| < 1 \qquad \text{with} \quad y_n(\pm 1) \qquad \text{regular.}$$

If $f(x) = \sum_{n=0}^{\infty} c_n y_n$, then give an expression for the constants c_n.
Show directly from the equation that

$$n(n+1)\int_{-1}^{1} y_n dx = 0 \qquad \text{and} \quad n(n+1)\int_{-1}^{1} x^2 y_n dx = 6\int_{-1}^{1} x^2 y_n dx - 2\int_{-1}^{1} y_n dx \ .$$

Deduce that the eigen-function expansion of x^2 has only two terms, $x^2 = c_0 y_0 + c_2 y_2$.

5. Show that the functions $U(x) \equiv J_n(\alpha x)$ and $V(x) \equiv J_n(\beta x)$ satisfy the equations

$$\frac{d}{dx}\left(x\frac{dU}{dx}\right) - \frac{n^2 U}{x} + \alpha^2 x U = 0 \qquad \text{and} \qquad \frac{d}{dx}\left(x\frac{dV}{dx}\right) - \frac{n^2 V}{x} + \beta^2 x V = 0 \ ,$$

where J_n is the Bessel function of order n which is finite at $x = 0$. By subtracting V times the first equation from U times the second, deduce that

$$(\beta^2 - \alpha^2)\int_0^a xJ_n(\alpha x)J_n(\beta x)dx = a\big[\alpha J_n(\beta a)J_n'(\alpha a) - \beta J_n(\alpha a)J_n'(\beta a)\big] \ .$$

Suppose $J_n(ka) = 0$. By setting $\alpha = k$ and considering the limit as $\beta \to k$, show that

$$\int_0^a xJ_n^2(kx)dx = \tfrac{1}{2}a^2\left[J_n'(ka)\right]^2 \ .$$

[This gives the normalisation factor for the eigen-functions $J_n(k_i a)$ considered in lectures.]