M2M2: Problem Sheet 6; Series solutions of PDEs

1. Find $u(x,t)$ such that $u_t = u_{xx}$ in $t > 0$, $0 < x < L$ subject to the initial condition $u(x,\,0) = x$ and the boundary conditions $u_x(0,\,t) = u_x(L,\,t) = 0$.

2. Find $u(x,t)$ such that $u_{tt} = u_{xx}$ in $t > 0$, $0 < x < L$ subject to the initial conditions

$$u(x,\,0) = 0 \qquad \text{for} \quad 0 < x < L \qquad \text{and} \quad u_t = \begin{cases} x & \text{for} \quad 0 < x < \frac{1}{2}L \\ L - x & \text{for} \quad \frac{1}{2}L < x < L \end{cases}$$

and the boundary conditions $u(0,\,t) = u(L,\,t) = 0$.

3. Obtain a series expansion of the solution $u(r,\,\theta)$ to the Dirichlet problem

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \text{in} \quad 0 < r < a\;, \quad 0 < \theta < 2\pi\;,$$

where $(r,\,\theta)$ are **plane** polar coordinates, with the boundary condition $u(a,\theta) = f(\theta)$. When $f(\theta) = \cos^2\theta$, show that $u = \frac{1}{2} + \frac{1}{2}(r/a)^2\cos(2\theta)$.

4. The function $u(r,\,\theta)$ in terms of **spherical** polar coordinates $(r,\,\theta,\,\phi)$ obeys Laplace's equation, $\nabla^2 u = 0$, in the spherical shell region $a < r < b$, with boundary conditions $u(a,\,\theta) = 0$ and $u(b,\,\theta) = \cos^3\theta$. Obtain an expression for u in terms of Legendre functions.

5. The top and bottom of a truncated circular cylinder are heated, so that the internal temperature, $T(r,\,z)$ in terms of **cylindrical** polar coordinates $(r,\,\theta,\,z)$, satisfies Laplace's equation,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \qquad \text{in} \quad 0 < r < a\;, 0 < z < 2a\;,$$

with the boundary conditions for some constant T_0

$$T(a,z) = 0 \qquad \text{for} \quad 0 < z < 2a \qquad \text{and} \quad T(r,0) = T(r,2a) = T_0 \qquad \text{for} \quad 0 < r < a\;.$$

Obtain the solution, for constants A_n to be determined,

$$T(r,\,z) = \sum_{n=1}^{\infty} A_n \cosh[k_n(z-a)]J_0(k_n r) \qquad \text{where} \quad J_0(k_n a) = 0\;.$$

(You may find the following identities useful: $xJ_0(x) = [xJ_1(x)]'$, $J_0'(x) = -J_1(x)$ and $\int_0^a r[J_0(k_n r)]^2\,dr = \frac{1}{2}a^2[J_1(k_n a)]^2$.)

6. An unusually shaped drum occupies the circular sector $0 < r < a$, $0 < \theta < \alpha$. When struck, its displacement $u(r,\,\theta,\,t)$ obeys the 2-D wave equation $u_{tt} = c^2\nabla^2 u$, together with the boundary condition $u = 0$ all round the boundary. Obtain the normal modes of the system (i.e. find the eigen-functions proportional to $\cos(\omega t)$.)