UNIVERSITY OF LONDON

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

BSc/MSci EXAMINATION (MATHEMATICS) MAY – JUNE 2003

This paper is also taken for the relevant examination for the Associateship

M3A10 Viscous Flow

DATE: 3rd June 2003 TIME: 10 am – 12 noon

Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Define the strain-rate tensor e_{ij}, and the stress tensor σ_{ij} for an incompressible Newtonian fluid with viscosity μ and density ρ.

Such a fluid occupies a fixed volume V, bounded by a stationary surface S with unit normal n_i acted upon by a body force F_i. Define E, the total kinetic energy within V, and starting from the Cauchy momentum equation

$$\rho \frac{Du_i}{Dt} = F_i + \frac{\partial \sigma_{ij}}{\partial x_j} \ ,$$

show that

$$\frac{dE}{dt} = \int_S u_i \, \sigma_{ij} n_j \, dS + \int_V u_i F_i \, dV - 2\mu \int_V e_{ij} e_{ij} \, dV \ ,$$

and interpret each term physically.

For a certain flow $e_{ij} = 0$ at every point within V. Express this relation in component form and show that

$$\frac{\partial u_1}{\partial x_1} = 0, \qquad \frac{\partial^2 u_1}{\partial x_2^2} = 0 = \frac{\partial^2 u_1}{\partial x_3^2} \qquad \text{and} \qquad \frac{\partial^2 u_1}{\partial x_2 \partial x_3} = 0 \ .$$

Using these results and symmetry arguments, deduce that

$$\mathbf{u} = \mathbf{U} + \boldsymbol{\Omega} \wedge \mathbf{x}$$

for spatially constant vectors $\mathbf{U}$ and $\boldsymbol{\Omega}$, and interpret this result physically.

2. Flow is driven in the rigid channel $a > y > -a$ by the oscillating pressure gradient

$$-\frac{\partial p}{\partial x} = G_0 + G_1 \cos \Omega t \ ,$$

where G_0, G_1 and Ω are constant. Show that a solution to the incompressible Navier-Stokes equations of the form $\mathbf{u} = (u(y,t),\, 0,\, 0)$ with

$$u = u_0(y) + \Re e \left[u_1(y) e^{i\Omega t} \right] \ ,$$

is possible, where $\Re e$ denotes the real part.

Find u_0 and show that for a suitable real constant δ,

$$u_1 = \frac{G_1}{\rho i \Omega} \left[1 - \frac{\cosh[(1+i)y/\delta]}{\cosh[(1+i)a/\delta]} \right] \ .$$

As $\Omega \to \infty$, show that the wall shear stress

$$\mu \left. \frac{\partial u}{\partial y} \right|_{y=a} \quad \to \quad -aG_0 - G_1 \left(\frac{\mu}{2\rho\Omega} \right)^{1/2} \left[\cos \Omega t + \sin \Omega t \right] \ .$$

Suppose $G_1 \gg G_0 > 0$. Discuss whether or not the velocity can be negative for some values of y and t.

3. In terms of cylindrical polar coordinates (r, θ, z), a paint-scraper occupies the half-plane $\theta = \frac{1}{2}\pi$ and moves with constant speed U parallel to a stationary table at $\theta = 0$. The quadrant $\frac{1}{2}\pi > \theta > 0$ contains very viscous paint. Starting from the unforced Stokes equations, show that the two-dimensional flow can be represented by

$$\nabla^2\omega = 0 \qquad \text{where} \quad \omega = -\nabla^2\psi \qquad \text{and} \quad \mathbf{u} = \left(\frac{1}{r}\frac{\partial\psi}{\partial\theta},\, -\frac{\partial\psi}{\partial r},\, 0\right) .$$

Write down the four boundary conditions to be applied at $\theta = 0, \frac{1}{2}\pi$ and find a solution

$$\omega = -r^{n-2}g(\theta) \qquad \text{and} \quad \psi = r^n f(\theta) \qquad \text{in} \quad 0 < \theta < \tfrac{1}{2}\pi ,$$

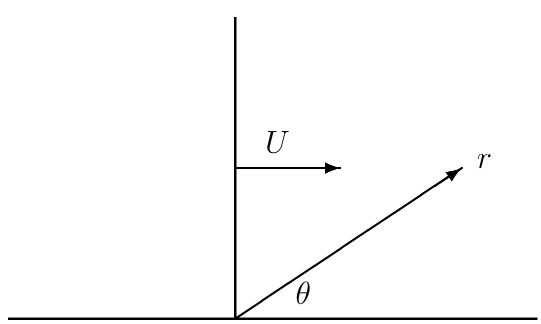
where n is a value suggested by the boundary conditions. Show that the tangential traction on the table surface,

$$\left.\frac{\mu}{r}\frac{\partial u_r}{\partial\theta}\right|_{\theta=0} = \frac{\pi\mu U}{\frac{1}{4}\pi^2 - 1}\,\frac{1}{r} .$$

What problem is encountered when evaluating the total stress per unit length in the z-direction exerted on a small length of table $0 < r < a$?

Suggest a reason for the difficulty, and a manner in which the model could be made more realistic.

$$\left[\text{In two-dimensions} \qquad \nabla^2\phi(r, \theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} .\right]$$



4. Write down the lubrication equations for a cylindrically symmetric flow

$$\mathbf{u} = (u_r(r,z),\ 0,\ u_z(r,z))$$

and state the conditions under which they can be expected to hold.

Two coaxial discs of radius a are at $z = \pm h(t)$, where $a \gg h$. Use lubrication theory to show that the pressure distribution between the discs is

$$p = p_0 + \frac{3\mu}{4h^3}\frac{dh}{dt}(r^2 - a^2),$$

stating any additional assumptions you make.

A constant, outwards normal force F is applied to each disc in an attempt to separate them. If initially the discs are a distance h_0 apart, show that the time required to pull them apart is

$$t = \frac{3\pi\mu a^4}{16 h_0^2 F}\ ,$$

and comment briefly on this result.

$$\left[\text{In axisymmetry,} \qquad \nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z}\ .\right]$$

5. A solid boundary at $y = 0$ has a slit in it at $x = 0$ through which incompressible fluid is sucked. The inviscid equations have the solution $\mathbf{u} = \nabla\phi$ where

$$\phi = -Q \ln r \qquad \text{and} \quad r^2 = x^2 + y^2 \ .$$

Calculate the slip velocity on $y = 0$ for $x > 0$ and write down the boundary layer equations and boundary conditions for this problem.

Seek a similarity solution with

$$\psi = A x^a f(\eta) \qquad \text{where} \quad \eta = B y x^b \ .$$

and show that $a = 0$ and $b = -1$. Choosing $A = -(Q\nu)^{1/2}$ and $B = (Q/\nu)^{1/2}$, deduce that

$$f''' + 1 - (f')^2 = 0 \ ,$$

and state the boundary conditions on f.

Verify that the function

$$f'(\eta) = 1 - \frac{6}{1+c} \qquad \text{where} \quad c = \cosh(\eta\sqrt{2} + \alpha) \ ,$$

solves the problem, provided that the constant α takes one of two values.

Sketch the velocity profile f' roughly in each case, and state which value of α is more plausible physically.