UNIVERSITY OF LONDON
IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

BSc/MSci EXAMINATION (MATHEMATICS) MAY – JUNE 2004
This paper is also taken for the relevant examination for the Associateship

M4A10 Viscous Flow (with advanced study)

DATE: Wednesday, 31 May 2004 TIME: 2.00 pm–4.00 pm

Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

QUESTION 1

$V(t)$ is a volume which deforms with the velocity $\mathbf{u}(\mathbf{x}, t)$ of a (possibly compressible) fluid, and $\theta(\mathbf{x}, t)$ is the density of some quantity associated with the fluid. By considering changes over a small time interval δt, prove that

$$\frac{d}{dt}\left(\int\limits_{V(t)} \theta(\mathbf{x}, t)\, dV\right) = \int\limits_{V(t)} \left(\frac{D\theta}{Dt} + \theta\nabla\cdot\mathbf{u}\right) dV \ ,$$

where D/Dt denotes the material derivative.

Deduce the equation for mass conservation, and derive the condition for incompressible flow.

The temperature (or heat density) of a fluid is denoted by $T(\mathbf{x}, t)$. Across any surface with unit normal $\widehat{\mathbf{n}}$, thermal conduction ensures that there is a heat flux in the $\widehat{\mathbf{n}}$-direction $-K\widehat{\mathbf{n}}\cdot\nabla T$, where K is a known constant. Deduce that T satisfies the equation

$$\frac{\partial T}{\partial t} + \nabla\cdot(T\mathbf{u}) = K\nabla^2 T \ .$$

Discuss whether or not it is possible for a flow to be simultaneously (a) steady, (b) incompressible and (c) for the density to vary with temperature according to the law

$$\rho = \rho_0 - \alpha T \qquad \text{where } \alpha \text{ and } \rho_0 \text{ are constant.}$$

QUESTION 2

An incompressible Newtonian fluid of density ρ and viscosity μ occupies the slab $0 < y < h$. The boundary $y = 0$ is stationary, but the boundary $y = h$ is accelerated so that its velocity is $(at,\, 0,\, 0)$ for some constant a. Show that a uni-directional solution $\mathbf{u} = (u_0(y) + tu_1(y),\, 0,\, 0)$ is possible, and find it assuming no pressure gradient acts in the x-direction.

Calculate the viscous stresses acting on the two boundaries $y = 0$ and $y = h$, and show that these change sign at $t = \rho h^2/(6\mu)$ and $t = -\rho h^2/(3\mu)$ respectively.

Verify that

$$\frac{d}{dt}\int_0^h \rho u(y,\, t)\, dy = \mu[u_y]_0^h \ ,$$

and interpret this relation physically.

QUESTION 3

When is it appropriate to use the Stokes equations to find the steady flow of an incompressible Newtonian fluid?

Such a flow occurs in a region V bounded by a surface S, so that

$$\nabla \cdot \mathbf{u} = 0, \qquad \frac{\partial \sigma_{ij}}{\partial x_j} = 0 \qquad \text{with} \quad \mathbf{u} = \mathbf{U}_0 \qquad \text{on} \quad S$$

for a given constant $\mathbf{U}_0$. If e_{ij} is the strain-rate tensor for this flow, and $\overline{e}_{ij}$ is the strain-rate tensor for a kinematically admissible flow field $\overline{\mathbf{u}}$, so that

$$\nabla \cdot \overline{\mathbf{u}} = 0 \qquad \overline{\mathbf{u}} = \mathbf{U}_0 \qquad \text{on} \quad S,$$

prove that

$$2\mu \int_V e_{ij} e_{ij} \, dV \leq 2\mu \int_V \overline{e}_{ij} \overline{e}_{ij} \, dV \ .$$

An irregularly shaped solid body moves slowly through the fluid with constant velocity $\mathbf{U}_0$ without rotating in response to a constant force $\mathbf{F}$. Interpret physically the relation

$$\mathbf{U}_0 \cdot \mathbf{F} = 2\mu \int_V e_{ij} e_{ij} \, dV \ ,$$

where V is the fluid region outside the body. Defining a suitable $\overline{\mathbf{u}}$, show that if one body would fit entirely within the volume occupied by a second body, then the force required to move the second with the same speed is larger.

A cube of side $2a$ falls slowly through a viscous fluid with speed U_0. Explain why the magnitude of the drag force must be $k\mu U_0 a$, where k is a dimensionless constant. Show that

$$6\pi < k < 6\pi\sqrt{3} \ ,$$

quoting any other result which you use.

QUESTION 4

An incompressible viscous flow occurs in a two-dimensional channel of length L and height h. Under what conditions are the equations $p_y = 0$, $p_x = \mu u_{yy}$ applicable?

The planar surface $y = 0$ moves with constant velocity $(U,\, 0,\, 0)$, past a stationary block with boundary

$$y = h(x) \equiv h_1 - \alpha x, \quad \text{for} \quad 0 < x < L$$

where h_1, L, and α are positive constants with $h_2 = h_1 - \alpha L > 0$.

Using lubrication theory, find $u(x,\, y)$, the x-component of the velocity in terms of the pressure gradient, and show that the pressure under the block is given by

$$p = p_1 + \frac{6\mu}{\alpha}\left[U\left(\frac{1}{h} - \frac{1}{h_1}\right) - Q\left(\frac{1}{h^2} - \frac{1}{h_1^2}\right)\right],$$

where Q is the volume flux of the fluid,

$$Q = \int_0^h u\, dy.$$

Assuming that $p = p_1$ at $x = L$ (as well as at $x = 0$), deduce that

$$p - p_1 = \frac{6\mu U}{\alpha}\, \frac{(h_1 - h)(h - h_2)}{h^2(h_1 + h_2)}\ .$$

Hence calculate the y-component of the total force acting on the surface $y = 0$, -$\int_0^L (p - p_1) dx$, and comment on any feature of physical importance.

QUESTION 5

Starting from the steady, incompressible Navier-Stokes equations, give a careful account of the arguments leading to the boundary layer equations

$$uu_x + vu_y = UU'(x) + \nu u_{yy} \ , \qquad u_x + v_y = 0$$

subject to the boundary conditions

$$u = v = 0 \quad \text{on } y = 0, \qquad u \to U(x) \quad \text{as } y \to \infty.$$

Derive conditions on a and b and $U(x)$ such that these equations might have solutions of the form $u = x^a f'(\zeta)$ where $\zeta = yx^b$. Deduce a differential equation and boundary conditions for $f(\zeta)$.