1. Explain the concept of a stress tensor for a general fluid. What property must the tensor possess if only standard volume forces act on the fluid?

Write down the mass conservation relation for a fluid of density $\rho(\mathbf{x},\, t)$ and velocity $\mathbf{u}(\mathbf{x},\, t)$.

For a **compressible** monatomic gas, the stress tensor is

$$\sigma_{ij} = -p\delta_{ij} + 2\mu(e_{ij} - \tfrac{1}{3}(\nabla\cdot\mathbf{u})\delta_{ij})$$

where $p(\mathbf{x},\, t)$ is the pressure, e_{ij} is the strain-rate tensor and μ is the constant viscosity. Starting from the Cauchy momentum equation

$$\rho\frac{Du_i}{Dt} = \frac{\partial\sigma_{ij}}{\partial x_j}$$

show that the momentum equation for the gas takes the same form as for an incompressible Newtonian fluid, provided the pressure is suitably modifed.

Stating carefully any results you use, obtain the energy balance equation over a deformable fluid volume $V(t)$ bounded by a surface S in the form

$$\frac{d}{dt}\int_V \tfrac{1}{2}\rho u_i u_i\, dV = \int_S u_i\tau_i\, dS - 2\mu\int_V \Phi\, dV + A$$

where $\Phi = e_{ij}e_{ij} - \frac{1}{3}(\nabla\cdot\mathbf{u})^2$ and A denotes a term which is absent for incompressible flows.

Identify the physical significance of each term, giving a plausible interpretation of the term A.

Calculate Φ for the spherically symmetric compression or expansion $u_1 = kx_1$, $u_2 = kx_2$, $u_3 = kx_3$ for constant k.

2. A stationary, solid plate occupies the plane $y = 0$.

Investigate steady, two-dimensional solutions of the incompressible Navier-Stokes equations given by a streamfunction of the form $\psi = xf(y)$. Obtain an equation for $f(y)$ and an expression for the pressure $p(x, y)$ in terms of f, given that $p(0, 0) = p_0$ and that $f'(y) \to A$ as $y \to \infty$, where A is a positive constant.

Write down the appropriate boundary conditions and find the flow in terms of the function $F(\xi)$ defined below, assuming $A > 0$.

If $A < 0$, no solution of this form exists. Suggest what is likely to happen in practice.

The function $F(\xi)$ is defined to be the solution to the problem:

$$F''' + FF'' - (F')^2 + 1 = 0 \qquad \text{in} \quad \xi > 0$$

$$\text{with} \quad F(0) = 0, \qquad F'(0) = 0, \qquad F'(\xi) \to 1 \qquad \text{as} \quad \xi \to \infty .$$

3. A needle of length l moves with speed U through fluid of viscosity μ at low Reynolds number. The drag is $2k\mu lU$ if the needle moves perpendicular to its axis and $k\mu lU$ if it moves parallel to its axis, where k is a dimensionless constant.

The needle is inclined at an angle α to the vertical and falls under gravity. Use the linearity of the Stokes equations to show that its velocity makes an angle β to the vertical where

$$\tan(\alpha-\beta) = \tfrac{1}{2}\tan\alpha \ .$$

Show that the maximum possible value of β is given by $\tan\beta = 1/(2\sqrt{2})$ and that this occurs when $\tan\alpha = \sqrt{2}$.

For this maximal value of β show that the speed of the needle is

$$U = \frac{mg}{\sqrt{2}k\mu l} \ ,$$

where mg is the needle's weight, adjusted for buoyancy.

Show that the total rate of energy dissipation in this case is

$$\tfrac{2}{3}\frac{(mg)^2}{kl\mu} \ .$$

[You are reminded that

$$\cos x = \frac{1}{(1+\tan^2 x)^{1/2}} \qquad \text{and} \qquad \tan(x-y) = \frac{\tan x - \tan y}{1+\tan x \tan y} \ .$$]

4. The axis of a circular cylinder of radius b is parallel to and a distance δ from the axis of a similar cylinder of radius a, where $a > b \gg \delta$. In terms of polar coordinates (r, θ) based on the centre of the larger cylinder, the gap between the two cylinders is $h(\theta)$. Show that

$$h(\theta) = a - b - \delta \cos\theta + O(\delta^2).$$

Fluid of density ρ and viscosity μ flows two-dimensionally in the thin gap between the two cylinders. The outer cylinder is stationary, but the inner one rotates with angular speed ω so that its surface speed is $b\omega$.

Explain the assumptions leading to the lubrication equations for the pressure p and the r- and θ- velocity components u and v:

$$p_r = 0, \qquad p_\theta = a\mu v_{rr} \qquad au_r + v_\theta = 0 \quad \text{in} \quad a > r > a - h(\theta)$$

and write down the appropriate boundary conditions. You may find it simpler to work with the variable $y = a - r$. Deduce that the pressure $p(\theta)$ satisfies

$$(h^3 p_\theta)_\theta = -6a^2 \mu\omega h_\theta \ .$$

Comment on the values of $p(-\pi)$ and $p(\pi)$ and hence find $dp/d\theta$, defining any integration constant in terms of definite integrals. Deduce that $dp/d\theta$ is an even function of θ.

The net force per unit length on the outer cylinder takes the vector form

$$\mathbf{F} = \int_{-\pi}^{\pi} p(\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} d\theta.$$

Using integration by parts, show that only one force component is non-zero.

5. Explain the circumstances in which the boundary layer equation

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = G(x) + \nu\psi_{yyy}$$

can be expected to hold, and give an account of its derivation from the steady, incompressible, Navier-Stokes equations.

Just outside the boundary layer on a solid plate at $y = 0$, the x-component of velocity is $u = U_0 e^{ax}$ where a is a positive constant. Seek a similarity solution in the boundary layer of the form $\psi = Ae^{cx}f(\eta)$ where $\eta = Bye^{bx}$ for suitable constants A, B, b and c. Obtain a third order ODE for $f(\eta)$ and give the appropriate boundary conditions.

Writing $w = f'(\eta)$ and $z = f(\eta)$, derive a second order ODE for $w(z)$. What are the appropriate boundary conditions now?