M3A10 Viscous Flow: Boundary Layer Theory

The Navier-Stokes equations behave well for small Reynolds number. It can be shown that a solution exists, and we proved that this solution was unique. Furthermore, the solutions are smooth and regular. At high Reynolds number the nonlinear $\mathbf{u} \cdot \nabla \mathbf{u}$ term gains in significance and the situation is very different. Existence is hard to prove, and there may be more than one possible solution. Smooth, steady symmetric flows may suddenly become unsteady and asymmetric for no obvious reason. Laminar flows may become unstable and **turbulence** may develop. Furthermore, singular regions may develop, especially near solid boundaries. These thin layers near boundaries are known as "boundary layers."

To understand why this occurs, consider the following simple ODE for $y(x)$ in terms of a small positive parameter ε

$$\varepsilon y''(x) + y'(x) = -1 \qquad \text{with} \quad y(0) = 0 = y(1) \; . \tag{1}$$

This problem has the solution

$$y = -x + \frac{1 + e^{-x/\varepsilon}}{1 + e^{-1/\varepsilon}} \simeq 1 - x + e^{-x/\varepsilon} \qquad \text{when} \quad \varepsilon \ll 1. \tag{2}$$

This function is essentially $1 - x$ except when $x \sim \varepsilon$. Over a layer of thickness $\sim \varepsilon$ near $x = 0$, the solution falls from 1 to 0. This is an example of a boundary layer. If we substitute $\varepsilon = 0$ in (2), we obtain a 1st order ODE, so we cannot impose two boundary conditions on it. The equation can only satisfy both boundary conditions by arranging for very high gradients near the boundary.

The Navier-Stokes equations behave similarly for very small viscosity (high Reynolds numbers). The viscosity μ multiplies the highest derivative in the equation, so that in the limit $\mu \to 0$, one of the boundary conditions cannot be satisfied. Inviscid flows may have slip velocities over solid walls. When the viscosity is small but non-zero we therefore anticipate that thin layers may develop near the walls across which the tangential velocity adjusts to zero fromits inviscid value. Let the layer thickness be δ, and let x and y be coordinates locally tangential and parallel to the wall, with typical scales of variation L (say) and δ. Let the corresponding velocity components u and v have typical scales U_0 and V_0 in the boundary layer. Then from the continuity condition

$$u_x + v_y = 0 \qquad \Longrightarrow \qquad V_0 \sim U_0 \delta / L \; ,$$

so that just as in lubrication theory, the normal velocity is small. Note that this requires that terms like uu_x and vu_y are of similar order. Normal derivatives in the viscous term will be much larger than the tangential derivatives, so that $\nu \nabla^2 u \simeq \nu u_{yy} \sim \nu U_0/\delta^2$.

We consider steady, 2-D flow. "Prandtl's Boundary Layer Hypothesis" states that in the layer the inertial terms and the viscous terms should balance. This suggests that the scale δ must be such that

$$uu_x \sim \nu u_{yy} \qquad \Longrightarrow \quad \frac{U_0^2}{L} \sim \frac{\nu U_0}{\delta^2} \qquad \Longrightarrow \quad \delta^2 \sim \frac{\nu L}{U_0}. \tag{3}$$

Now as y increases and we leave the boundary layer, we expect the viscous terms to become negligible. The flow should approach the inviscid flow solution, for which there is a slip velocity, i.e. $u \to U(x)$ as $y/\delta \to \infty$. This flow will have a pressure field associated with it with a scale $p_0 \sim \rho U_0^2$. We therefore assume that $p_x/\rho \sim uu_x$ in the x-momentum equation. However, p_y is a factor of (L/δ) larger than p_x, while uv_x is a factor of (δ/L) smaller than uu_x. It follows, just as for lubrication theory, that $p_y = 0$, so that the pressure does not vary across the layer. Putting all this together, we obtain the boundary layer equations

$$uu_x + vu_y = -\tfrac{1}{\rho}p_x + \nu u_{yy}, \qquad p_y = 0, \qquad u_x + v_y = 0\ . \tag{4}$$

The pressure gradient p_x can be evaluated just outside the layer where the viscous terms are negligible, and where $u = U(x)$. The boundary layer equations then take the form

$$uu_x + vu_y = UU'(x) + \nu u_{yy}\ , \qquad u_x + v_y = 0\ . \tag{5}$$

An appropriate set of boundary conditions are

$$u = v = 0 \quad \text{on} \quad y = 0, \qquad u \to U(x) \quad \text{as} \quad y \to \infty\ . \tag{6}$$

A single equation can be obtained by introducing a streamfunction $\psi(x, y)$ where $u = \psi_y$ and $v = -\psi_x$. Equation (5) then takes the form

$$\psi_y\psi_{xy} - \psi_x\psi_{yy} = UU'(x) + \nu\psi_{yyy}\ . \tag{7}$$

Notes on the boundary layer equations:

(1) Often it is convenient to nondimensionalise the problem. The pararmeter ν can be removed by rescaling ψ and y suitably. The boundary layer thickness $\delta = L(R_e)^{-1/2}$, where $R_e = U_0L/\nu$ is the Reynolds number.

(2) As $y \to \infty$ we cannot require that $v \to 0$. There is a small velocity ($\sim \delta$) flowing out of the boundary layer.

(3) The boundary layer equation (5) is **parabolic** in x and y, with x taking the part of the time-like variable (compare $u_t = u_{xx}$.) This means we must integrate downstream, in the direction of increasing x if $u > 0$. There is practically no upstream influence in boudary layers, whereas the full steady Navier-Stokes equations are **elliptic.**

(4) As we integrate the equations downstream, the equations can behave well, or they can develop a singularity. In the latter case, the boundary layer assumptions break down, and **the boundary layer separates.** This leads to the formation of a large wake, and completely alters the external flow structure.

(5) Generally speaking, separation will not occur while $UU' > 0$, a situation described as a 'favourable pressure gradient.' Once U begins to decrease, and the pressure gradient is unfavourable, then separation is likely. For example, irrotational flow around a cylinder predicts $U(x) = \sin(x)$, where x denotes length around the circumference of the cylinder. Now $UU' > 0$ until $x = \frac{1}{2}\pi$, when the surface slip-velocity begins to slow down. The layer separates off shortly after that point. The design of bodies intended to move at high Reynolds numbers (e.g. fish, aeroplanes) is governed by the need to inhibit or minimise separation of the boundary layer, a process called "streamlining."