M3A10: The energy equation

Let us now consider the kinetic energy of the fluid.

Taking, again, $V(t)$ as an arbitrary material volume, the kinetic energy of the fluid within V is

$$E = \int_V \tfrac{1}{2}\rho|\mathbf{u}|^2 dV. \tag{1.22}$$

Thus the rate of change of this energy is given by the Cauchy equation (1.19)

$$\frac{dE}{dt} = \int_V \rho u_i \frac{Du_i}{Dt}\, dV = \int_V u_i F_i \, dV + \int_V u_i \frac{\partial \sigma_{ij}}{\partial x_j}\, dV.$$

Now the final term here may be written

$$\begin{aligned}
\int_V u_i \frac{\partial \sigma_{ij}}{\partial x_j}\, dV &= \int_V \left[\frac{\partial}{\partial x_j}(u_i\sigma_{ij}) - \sigma_{ij}\frac{\partial u_i}{\partial x_j}\right] dV \\
&= \int_S u_i \sigma_{ij} n_j \, dS - \int_V \sigma_{ij} e_{ij}\, dV \\
&= \int_S u_i \tau_i \, dS - 2\mu \int_V e_{ij} e_{ij}\, dV.
\end{aligned}$$

We have used here the divergence theorem (0.8), the symmetry of the stress tensor (so that $\sigma_{ij}\Omega_{ij} \equiv 0$) and the incompressibility relation $\nabla \cdot \mathbf{u} \equiv e_{ii} = 0$. We have therefore shown that

$$\frac{dE}{dt} = \int_V u_i F_i \, dV + \int_S u_i \tau_i \, dS - 2\mu \int_V e_{ij} e_{ij}\, dV. \tag{1.23}$$

The first two terms on the right represent the rate of working by body and surface forces on the fluid within V, and the final term is therefore the rate of energy dissipation due to viscosity, which we can think of as a kind of friction. The rate of viscous heating Φ per unit volume is thus

$$\Phi = \sigma_{ij} e_{ij} = 2\mu e_{ij} e_{ij}.$$

The second law of thermodynamics demands that Φ and hence μ must be positive.

This heating can change the temperature T in the fluid. If then the density or viscosity depend on temperature, a further equation involving convection and diffusion of heat, is needed to determine $T(\mathbf{x}, t)$. We shall not pursue this (interesting) complication in this course.

Boundary Conditions

In order to determine the velocity $\mathbf{u}(\mathbf{x}, t)$ and pressure $p(\mathbf{x}, t)$ in some region V, we need to know what boundary conditions to apply on the surface S. The appropriate conditions to apply are that the velocity and the total stress should be continuous across any interface. In this course 'total stress' is just the fluid stress σ_{ij} and possibly **surface tension** (see below), but there could also be electromagnetic stresses, for example.

(a) Fluid/solid boundaries: Here, it is sufficient to require that the fluid velocity $\mathbf{u}$ be the same as the velocity of the boundary, so that for a stationary boundary

$$\mathbf{u} = 0 \; . \tag{1.24}$$

Note that (1.24) requires that the tangential velocity components be zero as well as the normal component. In **inviscid fluid mechanics** (M3A2) only the normal velocity need be continuous at an interface, and a 'slip velocity' must be permitted. The presence in the Navier-Stokes equation of the second derivative $\mu\nabla^2\mathbf{u}$ requires an extra boundary condition.

A solid boundary is able to provide whatever stress is needed to support the fluid motion. The continuity in stress then enables us to calculate the force on the solid due to the fluid.

(b) Fluid/fluid boundaries: These are more complicated, because the interface can move. Furthermore, it is a physical fact that an extra normal stress, due to **surface tension**, acts on the interface. This extra stress takes the form $\gamma K(\mathbf{x})$ where γ is the positive surface tension constant, and K is the curvature of the fluid surface, which can be defined by $K = \nabla \cdot \mathbf{n}$ where $\mathbf{n}$ is the unit normal to the interface.

If one of the fluids is dynamically negligible, as often happens with a liquid/gas interface, then we can treat one fluid as having a constant pressure p_0 and neglect its motion. If such a surface is stationary, then the appropriate boundary conditions to apply on the other fluid are zero normal velocity and zero tangential stress. (If the normal stress does not balance then the surface will not be stationary). More generally, if we describe the surface position at time t by the function $\zeta(\mathbf{x}, t) = 0$, then the boundary conditions can be written

$$\frac{D\zeta}{Dt} = 0 \qquad \text{and} \qquad \sigma_{ij} n_j = -(\gamma K + p_0) n_i \; . \tag{1.25}$$

In M3A2, $\mu = 0$ and the tangential stress condition is trivial.