M3A10: §1.1 Kinematics and mass conservation

Continuum hypothesis

We assume that at every point $\mathbf{x}$ of the fluid, and at all times t, we can define properties like density $\rho(\mathbf{x}, t)$, velocity $\mathbf{u}(\mathbf{x}, t)$, and (see later) pressure $p(\mathbf{x}, t)$, and that these vary smoothly (differentiably) over the fluid. Note that we do not deal with the dynamics of individual molecules. A small volume δV thus has mass $\delta V \rho$ and momentum $\delta V \rho \mathbf{u}$.

Time derivatives

A *fluid particle*, sometimes called a *material element*, is one that moves with the fluid, so that its velocity is $\mathbf{u}(\mathbf{x}, t)$ and its position $\mathbf{x}(t)$ satisfies

$$\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t).$$

The rate of change of a quantity as seen by a fluid particle is called the *material derivative* and written D/Dt. It is given by the chain rule as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla. \tag{1.1}$$

In particular, the *acceleration* of a fluid particle is

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}. \tag{1.2}$$

Reynolds' Transport theorem

Fluid volumes deform in time as the fluid moves. If $\theta(\mathbf{x}, t)$ is the density of some quantity associated with the fluid, the time evolution of the amount of that quantity in a fluid volume $V(t)$ is

$$\frac{d}{dt}\left[\int_{V(t)} \theta(\mathbf{x}, t)\, dV\right] = \int_{V(t)} \left(\frac{D\theta}{Dt} + \theta\nabla\cdot\mathbf{u}\right) dV\ . \tag{1.3}$$

This theorem was proved in lectures.

Mass conservation

Because matter is neither created nor destroyed, if we substitute $\theta = \rho$ in the transport theorem, the LHS is zero. Then, as the volume $V(t)$ is arbitrary, we see that the mass density $\rho(\mathbf{x}, t)$ satisfies

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{u}\,, \tag{1.4}$$

or equivalently

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.$$

The quantity $\rho\mathbf{u}$ is called the *mass flux*. For an *incompressible* fluid, the density of each material element is constant, and

$$\frac{D\rho}{Dt} = 0.$$

It follows that for **incompressible fluids**

$$\boxed{\nabla \cdot \mathbf{u} = 0\ . \qquad (1.5)}$$

We shall in this course restrict attention to fluids that are incompressible and have uniform density, so that ρ is independent of both $\mathbf{x}$ and t.

Streamfunctions in 2D and axisymmetry

For two-dimensional flows, the condition $\nabla \cdot \mathbf{u} = 0$ is automatically satisfied by

$$\mathbf{u} = \nabla \wedge (0, 0, \psi(x, y)) = (\psi_y, -\psi_x, 0). \tag{1.6}$$

$\psi(x, y)$ is called the *streamfunction*.

In axisymmetric flows, in terms of **cylindrical** polar coordinates $(r,\ \theta,\ z)$, the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ is satisfied by writing

$$\mathbf{u} = \nabla \wedge (0,\ \frac{\psi}{r},\ 0) = \left(-\frac{1}{r}\frac{\partial \psi}{\partial z},\ 0,\ \frac{1}{r}\frac{\partial \psi}{\partial r}\right)\ . \tag{1.7}$$

Here $\psi(r, z)$ is known as the Stokes streamfunction.

Alternatively, we can use **spherical** polar coordinates $(r,\ \theta,\ \phi)$. Recall that r and θ mean different things in spherical and cylindrical polars, and that ϕ is now the azimuthal angle, not θ. Now we write

$$\mathbf{u} = \nabla \wedge \left(0,\ 0,\ \frac{\psi}{r \sin\theta}\right) = \left(\frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta},\ -\frac{1}{r \sin\theta}\frac{\partial \psi}{\partial r},\ 0\right)\ . \tag{1.8}$$

The Stokes streamfunction $\psi(r,\ \theta)$ is the same as in cylindrical polars, but written in different coordinates.