M3A10: Problem Sheet 1: The Navier-Stokes Equations

On this sheet ρ and μ denote the density and viscosity of a Newtonian fluid.

1. Show that the simple shear flow $\mathbf{u} = (ay,\, 0,\, 0)$ for some constant a can be written as the sum of a solid body rotation and a two-dimensional strain. What must the pressure distribution be to maintain this flow in the absence of any body forces?
If the flow occurs between two planar plates at $y = 0$ and $y = h$, calculate the force per unit area on each plate.

2. A two-dimensional flow field is defined in Cartesian coordinates by $\mathbf{u} = (\psi_y,\, -\psi_x,\, 0)$ where the streamfunction

$$\psi = a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3 \qquad \text{for constant } a_0,\, a_1,\, a_2,\, a_3\,.$$

Show that the viscous part of the total surface force acting on any fluid region of volume V is $\mu V(2a_1 + 6a_3,\, -6a_0 - 2a_2,\, 0)$. Hence show that this force vanishes if the flow is irrotational (i.e. if $\boldsymbol{\omega} \equiv \nabla \wedge \mathbf{u} = 0$ everywhere.)

3. Show that for a volume V with a **stationary** rigid boundary, an alternative form for the rate of energy dissipation Φ is

$$\Phi \equiv 2\mu \int_V e_{ij}e_{ij}\, dV = \mu \int_V \omega^2\, dV\ ,$$

where $\omega = |\boldsymbol{\omega}|$. What happens if the vorticity $\boldsymbol{\omega} = 0$ throughout V?

4. If we want to work in non-Cartesian coordinate systems then we can use (0.5) and (0.6) from the printed sheet to rewrite the $\mathbf{u}\cdot\nabla\mathbf{u}$ and $\nabla^2\mathbf{u}$ terms in the new coordinates. For **Cylindrical Polar Coordinates** $(r,\, \theta,\, z)$, for example, the scale factors (recall M2M1 and M3A2) are $h_r = 1$, $h_\theta = r$, $h_z = 1$. Writing $\mathbf{u} = (u_r,\, u_\theta,\, u_z)$, show that

$$\boldsymbol{\omega} \equiv \nabla \wedge \mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z},\ \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r},\ \frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\ .$$

Specialising to **axi-symmetry**, so that $\partial\mathbf{u}/\partial\theta \equiv 0$, derive the Navier-Stokes equations in these coordinates in the form

$$\begin{aligned}
\rho\left(\frac{Du_r}{Dt} - \frac{u_\theta^2}{r}\right) &= -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \left[\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right]\right)\\
\rho\left(\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r}\right) &= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \left[\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right]\right)\\
\rho\frac{Du_z}{Dt} &= -\frac{\partial p}{\partial z} + \mu\nabla^2 u_z
\end{aligned}$$

where the terms in square brackets are absent in axisymmetry, but are given here for completeness. Here

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u_r\frac{\partial f}{\partial r} + \left[u_\theta\frac{1}{r}\frac{\partial f}{\partial \theta}\right] + u_z\frac{\partial f}{\partial z} \qquad \text{and} \qquad \nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \left[\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}\right] + \frac{\partial^2 f}{\partial z^2}\ .$$