M3A10: Problem Sheet 2: Exact solutions of the Navier-Stokes Equations

On this sheet $\nu = \mu/\rho$ is the kinematic viscosity of an incompressible Newtonian fluid.

1. Steady flow is driven down a channel in $0 < y < h$ by a pressure gradient $G_0 = -dp/dx$. The walls are porous, and fluid is sucked in the y-direction, so that $\mathbf{u} = (0,\, V,\, 0)$ on both $y = 0$ and $y = h$. Seek a solution with $\mathbf{u} = (u(y),\, v(y),\, 0)$ and show that

$$u = \frac{G_0 h}{\rho V}\left(\frac{y}{h} - \frac{1 - e^{Vy/\nu}}{1 - e^{Vh/\nu}}\right) .$$

Sketch $u(y)$ for the two cases $Vh/\nu \ll 1$ and $Vh/\nu \gg 1$, assuming $V > 0$.

2. Show that in terms of cylindrical polar coordinates $(r,\, \theta,\, z)$ a circular flow of the form $\mathbf{u} = (0,\, u_\theta(r,\, t),\, 0)$ is possible, provided [Use question 4 on sheet 1]

$$\frac{\partial (r u_\theta)}{\partial t} = \nu\left(\frac{\partial^2 (r u_\theta)}{\partial r^2} - \frac{1}{r}\frac{\partial (r u_\theta)}{\partial r}\right) .$$

Deduce that $u_\theta = A/r$ is a solution except for a singularity on the axis $r = 0$. [This flow is due to a **line vortex.**]

Seek a **similarity solution** $r u_\theta(r,\, t) = f(\eta)$ where $\eta = r/(\nu t)^{1/2}$, such that u_θ is finite on $r = 0$ for $t > 0$, but $u_\theta = A/r$ at $t = 0$. Write down the equation and boundary conditions satisfied by $f(\eta)$ and deduce that

$$u_\theta = \frac{A}{r}\left(1 - e^{-r^2/(4\nu t)}\right) .$$

Describe the flow for $r \ll \sqrt{\nu t}$ and for $r \gg \sqrt{\nu t}$.

3. Convince yourself that the kinematic viscosity ν and a steamfunction ψ have the same physical dimensions.

Steady, two-dimensional, radial flow in a wedge-shaped corner region is given in terms of cylindrical polar coordinates $(r,\, \theta,\, z)$ by $\mathbf{u} = \nabla \wedge (0,\, 0,\, \psi)$ where $\psi = \nu f(\theta)$. Show that

$$f'''' + 2f'f'' + 4f'' = 0 .$$

Integrate this equation twice to show that

$$(f'')^2 = Af' + B - 4(f')^2 - \tfrac{2}{3}(f')^3 \qquad \text{for arbitrary constants } A \text{ and } B.$$

Verify that $f = 6/\theta - 2\theta$ is a possible solution, and sketch the flow for $1 < \theta < 2$.