M3A10: Problem Sheet 3: Low Reynolds number flows

1. If $\mathbf{v}(\mathbf{x})$ is a vector harmonic function, so that $\nabla^2\mathbf{v} = 0$, show that

$$\mathbf{u} = 2\mathbf{v} - \nabla(\mathbf{v}\cdot\mathbf{x}) \qquad \text{with} \quad p = -2\mu\nabla\cdot\mathbf{v}$$

satisfy the unforced Stokes equations.

2. Prove the **reciprocal theorem**, namely that if two unforced Stokes flows $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ with corresponding stresses $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$ can occur in a region V bounded by a surface S, then

$$\int_S u_i^{(1)}\tau_i^{(2)}\,dS = \int_S u_i^{(2)}\tau_i^{(1)}\,dS \ ,$$

where the tractions are defined by $\tau_i = \sigma_{ij}n_j$.

3. Two tiny spherical bubbles of radius a are rising through unbounded fluid under gravity. The separation between their centres, d, is much larger than their radius.

By assuming the fluid around one bubble is moving with a constant velocity due to the far field of the other bubble, estimate the rise velocity if (a) the two bubbles are vertically above one another and (b) if the two bubbles are in the same horizontal plane.

Now suppose three identical bubbles are rising and initially they are equidistant and in a vertical column. Again, only considering effects first order in (a/d), describe what will happen subsequently.

4. Two dimensional Stokes flow, with $\mathbf{u} = \nabla\wedge(0,\,0,\,\psi(r,\theta))$ in terms of cylindrical polar coordinates $(r,\,\theta,\,z)$, takes place in the corner $-\alpha < \theta < \alpha$. Seek a solution with $\psi = r^{n+1}f(\theta)$, where n is not necessarily an integer, subject to no-slip boundary conditions. Show that for a solution which is **even** in θ to exist we must have $\sin 2n\alpha = -n\sin 2\alpha$, while an **odd** solution requires $\sin 2n\alpha = n\sin 2\alpha$. By considering the graph of $\frac{\sin x}{x}$ show that no real solutions for n exist for some values of α [note $n = 0,\,\pm 1$ don't work.]

5. Find the form of $f(r)$ such that the Stokes equations have the solution $\mathbf{u} = (0,\,0,\,u_\phi)$ with $u_\phi = f(r)\sin\theta$ in terms of spherical polar coordinates $(r,\,\theta,\,\phi)$, with zero pressure.

A sphere of radius a rotates with constant angular speed Ω in fluid at rest at infinity. Show that $\mathbf{u} = (0,\,0,\,u_\phi)$ where

$$u_\phi = \Omega a^3\frac{\sin\theta}{r^2}$$

Given that in these coordinates the r-ϕ strain component is

$$e_{r\phi} = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right) \ ,$$

show that the total moment of the viscous forces on the sphere has magnitude $8\pi\mu a^3\Omega$.