M3A10: Sheet 4: Flows in thin layers: Lubrication and Boundary Layers.

1. A thin layer of honey of thickness $h(x,t)$ adheres to a vertical knife, where x measures distance down the knife. On the free surface $y = h$, the pressure is constant, and $\partial u/\partial y = 0$ Neglecting surface tension, show that h satisfies

$$\frac{\partial h}{\partial t} + \frac{\rho g}{3\mu}\frac{\partial (h^3)}{\partial x} = 0 \ .$$

Earlier in the course we obtained the solution with h constant. Find here a similarity solution with $h = f(xt^a)$.

2. The wind blows perpendicular to a vertical wall at one end of a large reservoir, exerting a constant stress τ on the water surface $y = 0$, so that

$$\mu\frac{\partial u}{\partial y} = \tau \qquad \text{on} \quad y = 0$$

Assuming the water surface remains flat, and that motion occurs in a boundary layer of thickness $\delta(x)$, derive a similarity solution for the steady flow, with

$$\psi = U(x)\delta(x)f(\eta) \qquad \text{where} \quad \eta = y/\delta(x) \qquad \text{and} \quad \delta \sim x^n.$$

Using the above stress condition and balancing viscous and inertial terms, obtain the form of $U(x)$ and a differential equation for $f(\eta)$. What are the appropriate boundary conditions?

3. A thin jet emerges from the origin along the x-axis into quiescent fluid. Use boundary layer arguments to show that in the jet

$$uu_x + vu_y = \nu u_{yy} \qquad \text{and prove that} \quad \int_{-\infty}^{\infty} u^2\, dy = M \qquad \text{is constant.}$$

Deduce that a similarity solution of the form

$$\psi = x^a f(yx^b) \qquad \text{requires} \quad a = \tfrac{1}{3}, \quad b = -\tfrac{2}{3} \ .$$

Using the substitution

$$\psi = (M\nu x)^{1/3}F(\eta) \qquad \text{where} \quad \eta = y[M/(\nu x)^2]^{1/3} \ ,$$

show that

$$F''' + \tfrac{1}{3}(F'^2 + FF'') = 0 \ .$$

Integrate this equation a few times and find the value of A such that $F = A\tanh(\frac{1}{6}A\eta)$ solves the problem. Sketch the velocity profile.