3. Low Reynolds number: Stokes flows

The Reynolds number

When the nonlinear advection term $\mathbf{u} \cdot \nabla \mathbf{u} = 0$, we can often solve the equations. Sometimes, though $\mathbf{u} \cdot \nabla \mathbf{u} \neq 0$, it may nevertheless be small compared to the viscous term $\nu \nabla^2 \mathbf{u}$.

To quantify the relative sizes of the inertial and viscous terms, we introduce the idea of scales of variation. Let U_0 be a typical value of $|\mathbf{u}|$, and let L be a typical length scale on which $\mathbf{u}$ is varying. Then we estimate

$$\frac{||\rho \mathbf{u} \cdot \nabla \mathbf{u}||}{||\mu \nabla^2 \mathbf{u}||} \simeq \frac{\rho U_0^2/L}{\mu U_0/L^2} = \frac{U_0 L}{\nu} = R_e \; , \tag{3.1}$$

which we call **the Reynolds number** of the flow. It is clear there is some flexibility about the choice of U_0 and L and hence the precise definition of R_e. But if the Reynolds number is high ($R_e \gg 1$) viscous forces may be negligible, as in M3A2, whereas at low Reynolds number ($R_e \ll 1$) the nonlinear inertial forces may be ignored. This is the limit to be considered in this chapter.

The Stokes equations

In this case $R_e \ll 1$ and we neglect inertial terms in the Navier-Stokes equations (1.20) to obtain

$$\mu \nabla^2 \mathbf{u} = \nabla p - \widetilde{\mathbf{F}} \qquad \nabla \cdot \mathbf{u} = 0. \tag{3.2}$$

These are the (forced) Stokes equations. Natural boundary conditions are that at each point of S either $\mathbf{u}$ or the traction $\sigma_{ij} n_j$ is given. Equation (3.2) is equivalent to

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\widetilde{F}_i \qquad \sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij} \qquad \nabla \cdot \mathbf{u} = 0. \tag{3.3}$$

Instantaneity

There are no time derivatives in (3.2). Thus $\mathbf{u}$ responds instantaneously to the boundary motion and the force $\widetilde{\mathbf{F}}$. For instance, a sphere falling in an unbounded fluid achieves its terminal velocity at once.

Linearity

There is no $\mathbf{u}\cdot\nabla\mathbf{u}$ term in (3.2); therefore $\mathbf{u}$, p and σ_{ij} are linearly forced by any boundary motion or body force. If for instance we have a falling sphere, doubling the velocity will double σ_{ij} and thus double the drag. More generally, force $\propto$ velocity rather than acceleration.

Consider the drag force $-\mathbf{F}$ acting on a solid body moving with velocity $\mathbf{U} = (U_1, U_2, U_3)$. Because of the linearity of the problem, we can say $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$ where $\mathbf{F}_1$ is the force when the body moves with velocity $(U_1, 0, 0)$ and similarly for $\mathbf{F}_2$ and $\mathbf{F}_3$. Now let the body be a cube aligned with the coordinate axes. By symmetry, $\mathbf{F}_1 = (\alpha U_1, 0, 0)$ for some constant α. Similarly, $\mathbf{F}_2 = (0, \alpha U_2, 0)$ for the same constant α and hence in general $\mathbf{F} = \alpha\mathbf{U}$. Surprisingly, the Stokes drag force for a cube is the same regardless of its orientation. The same is not true at higher Reynolds numbers.

Reversibility

If the velocity on the boundary of a Stokes flow is reversed then so is the velocity everywhere in the fluid. If a prescribed boundary motion is reversed over time then each material point retraces its history. This will be beautifully illustrated in a video.

Does a sphere falling by a wall migrate towards or away from the wall? Neither: on reversal of $\mathbf{g}$, $\mathbf{u}$ must reverse and so if the sphere were to move towards the wall under $\mathbf{g}$ then it would move away from the wall under $-\mathbf{g}$.

Uniqueness Theorem for Stokes Flows

Theorem: There exists at most one Stokes flow in a volume V for which $\mathbf{u}$ is specified on the boundary.

Proof: Suppose $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are two such flows. Let $\mathbf{u}^* = \mathbf{u}^{(1)} - \mathbf{u}^{(2)}$, $\sigma^*_{ij} = \sigma^{(1)}_{ij} - \sigma^{(2)}_{ij}$ and $e^*_{ij} = e^{(1)}_{ij} - e^{(2)}_{ij}$. Then $\mathbf{u}^* = 0$ on S while (3.3) gives that $\frac{\partial \sigma^*_{ij}}{\partial x_j} = 0$ and $\frac{\partial u^*_i}{\partial x_i} = 0$. Now consider

$$2\mu \int_V e^*_{ij} e^*_{ij}\, dV = \int_V \sigma^*_{ij} \frac{\partial u^*_i}{\partial x_j}\, dV = \int_V \frac{\partial}{\partial x_j}\left(\sigma^*_{ij} u^*_i\right)\, dV = \int_S \sigma^*_{ij} u^*_i n_j\, dS = 0.$$

Thus since $e^*_{ij} e^*_{ij} \geq 0$ we must have $e^*_{ij} = 0$, so that for example $\partial u^*_1/\partial x_1 = 0$. But since $u^*_1 = 0$ on S we have $u^*_1 = 0$ and hence $\mathbf{u}^* = 0$ everywhere.

A more sophisticated argument (including the $\mathbf{u}\cdot\nabla\mathbf{u}$ term) proves uniqueness if $R_e < \pi\sqrt{3}$, but for large R_e there may be more than one solution.

Theorem: Minimum dissipation

Theorem: Suppose $\mathbf{u}(\mathbf{x})$ is the unique Stokes flow in V satisfying $\mathbf{u} = \mathbf{u}_0(\mathbf{x})$ on S. Let $\overline{\mathbf{u}}(\mathbf{x})$ be another 'kinematically possible' flow in V such that $\nabla \cdot \overline{\mathbf{u}} = 0$ and $\overline{\mathbf{u}} = \mathbf{u}_0$ on S. Then

$$2\mu \int_V \overline{e}_{ij}\overline{e}_{ij}\, dV \geq 2\mu \int_V e_{ij}e_{ij}\, dV, \tag{3.4}$$

with equality only if $\mathbf{u} = \overline{\mathbf{u}}$, i.e. energy dissipation is minimum in Stokes flow.

Proof: Let $\mathbf{u}^* = \mathbf{u} - \overline{\mathbf{u}}$ and $e^*_{ij} = e_{ij} - \overline{e}_{ij}$, so that $\mathbf{u}^* = 0$ on S and $e^*_{ii} = 0$. Consider

$$\int_V \left(\overline{e}_{ij}\overline{e}_{ij} - e_{ij}e_{ij}\right) dV = -\int_V e^*_{ij}\left(\overline{e}_{ij} + e_{ij}\right) dV = \int_V e^*_{ij}e^*_{ij}\, dV - 2\int_V e^*_{ij}e_{ij}\, dV$$

The first term is clearly positive; we now show the last term is zero.

$$2\mu \int_V e^*_{ij}e_{ij}\, dV = \int_V \sigma_{ij}e^*_{ij}\, dV = \int_V \frac{\partial u^*_i}{\partial x_j}\sigma_{ij}\, dV = \int_S u^*_i\sigma_{ij}n_j\, dS = 0.$$

As an example of this theorem, we consider the drag $\mathbf{F}$ on a rigid particle of arbitrary shape moving with velocity $\mathbf{U}$ in unbounded fluid. The rate of working of this force is

$$U_iF_i = \int_S u_i\sigma_{ij}n_j\, dS = 2\mu \int_V e_{ij}e_{ij}\, dV \leq 2\mu \int_V \overline{e}_{ij}\overline{e}_{ij}\, dV$$

where $\overline{e}_{ij}$ is the strain rate for any kinematically admissable flow field. If we choose $\overline{\mathbf{u}} = \mathbf{U}$ inside a region $\overline{V}$ enclosing the body and $\overline{\mathbf{u}}$ to be a Stokes flow outside $\overline{V}$, then $\overline{e}_{ij} = 0$ inside $\overline{V}$, while the RHS is the rate of working of the drag of a solid body occupying $\overline{V}$. It follows that the drag in Stokes flow on any body is less than the drag of any larger body.

The biharmonic equation

Taking the divergence of the Stokes equations (3.2) (with $\tilde{\mathbf{F}} = 0$) we see that p is harmonic, $\nabla^2 p = 0$. Taking the curl we see similarly that the vorticity vector is harmonic, $\nabla^2 \boldsymbol{\omega} = 0$.

$$\nabla^2 p = 0 \qquad \text{and} \qquad \nabla^2 \boldsymbol{\omega} = 0 \tag{3.5}$$

For a planar flow $\mathbf{u} = \nabla \times (0, 0, \psi(x, y))$ and $\boldsymbol{\omega} = (0, 0, -\nabla^2\psi)$. Thus

$$\nabla^2(\nabla^2\psi) = \nabla^4\psi = 0 \tag{3.6}$$

so that ψ satisfies the **biharmonic equation.** We can first solve $\nabla^2\omega = 0$ and then $\nabla^2\psi = -\omega$.

A similar result holds in axisymnmetry. If R is the distance from the axis of symmetry, the Stokes steamfunction ψ is found to obey the equation

$$D^2(D^2\psi) = 0 \qquad \text{where} \quad D^2\Psi = R^2\nabla \cdot \left(\frac{\nabla\psi}{R^2}\right) . \tag{3.7}$$

Stokes flow due to a translating sphere

We consider the inertialess flow generated by a sphere of radius a and velocity $\mathbf{u}$ immersed in unbounded fluid of viscosity μ which is at rest at infinity. In particular we want to calculate the force $\mathbf{F}$ exerted by the sphere on the fluid.

The linearity of the Stokes equations requires that F is proportional to both U and μ. Dimensional arguments therefore give $F = \alpha\mu a U$, where α is a positive dimensionless constant. The isotropy of the sphere's shape then implies that $\mathbf{F} = \alpha\mu a\mathbf{U}$. We must do some work to find the constant α.

We take spherical polars (r, θ, ϕ) with $\theta = 0$ parallel to $\mathbf{u}$. The flow is then axisymmetric with no ϕ dependence and so admits a Stokes streamfunction $\psi(r, \theta)$ such that the components of $\mathbf{u}$ are (see (1.8))

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta} \qquad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}.$$

It follows from the Stokes equations that $D^2\left(D^2\psi\right) = 0$ as in (3.7), where

$$D^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right). \tag{3.8}$$

The no-slip condition on the sphere surface $\mathbf{u} = (U\cos\theta,\ -U\sin\theta,\ 0)$ gives

$$\psi = \frac{1}{2}Ua^2\sin^2\theta \quad \text{and} \quad \frac{\partial\psi}{\partial r} = Ua\sin^2\theta \quad \text{on } r = a.$$

Finally, for $r \to \infty$, $\psi = o(r^2)$. We look for a solution $\psi = f(r)\sin^2\theta$ so that

$$\Omega = -D^2\psi = F(r)\sin^2\theta \qquad \text{where} \quad F(r) = f'' - \frac{2f}{r^2} \qquad \text{and} \quad F'' - \frac{2F}{r^2} = 0$$

as $D^2\Omega = 0$. Solving for F and f we have

$$f = Ar^4 + Br^2 + Cr + \frac{D}{r}$$

and the boundary conditions

$$f(a) = \tfrac{1}{2}Ua^2, \quad f'(a) = Ua, \quad f'' \to 0 \text{ as } r \to \infty$$

give $A = B = 0$, $C = \frac{3}{4}Ua$ and $D = -\frac{1}{4}Ua^3$. Substituting back we obtain

$$\psi = \frac{Ua^2}{4}\left(\frac{3r}{a} - \frac{a}{r}\right)\sin^2\theta \qquad \text{and} \quad \Omega = \tfrac{3}{2}U\frac{a}{r}\sin^2\theta \tag{3.9}$$

and

$$u_r = 2\left(\frac{C}{r} + \frac{D}{r^3}\right)\cos\theta \quad \text{and} \quad u_\theta = \left(-\frac{C}{r} + \frac{D}{r^3}\right)\sin\theta. \tag{3.10}$$

We can also obtain the pressure (to within an arbitrary constant, p_∞) from

$$\nabla p = -\mu\nabla \wedge \boldsymbol{\omega} = -\mu\nabla \wedge \left(0,\, 0,\, \frac{\Omega}{r\sin\theta}\right) = \left(\frac{-\mu}{r^2\sin\theta}\frac{\partial\Omega}{\partial\theta},\, \frac{\mu}{r\sin\theta}\frac{\partial\Omega}{\partial r},\, 0\right)$$

so that

$$p - p_\infty = \frac{2C\mu\cos\theta}{r^2}. \tag{3.11}$$

The stress can now be determined from (3.2) although care must be taken in evaluating e_{ij} in this curvilinear co-ordinate system. The normal $\mathbf{n}$ is in the $(-r)$-direction, and the traction components in the r and θ-directions are

$$\sigma_{rr} = -p + 2\mu\frac{\partial u_r}{\partial r}\ , \qquad \sigma_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right)\ . \tag{3.12}$$

By symmetry, the net force exerted by the sphere on the fluid must point in the z-direction, so that

$$|\mathbf{F}| = -\int_S (\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta)\, dS.$$

Finally, we obtain the force on the fluid as

$$\mathbf{F} = 6\pi\mu a\mathbf{u}, \tag{3.13}$$

a result known as Stokes' law.

The flow at infinity; Stokeslets

Note the fore and aft symmetry in the streamline pattern at low R_e. The velocity decays slowly as $r \to \infty$, $\mathbf{u} \sim \frac{1}{r}$, so that far field effects are important and distant boundaries and other particles may affect the flow.

We can calculate $\mathbf{F}$ more easily by moving the integral to a sphere at infinity using the divergence theorem;

$$F_i = \int_{r=a} \sigma_{ij} n_j \, dS = -\int_{S_\infty} \sigma_{ij} n_j, dS.$$

Only terms of order r^{-2} in σ_{ij} (r^{-1} in $\mathbf{u}$ or r in ψ) therefore contribute to the force. In the far field $\psi \sim Cr\sin^2\theta$ and $p - p_\infty \sim 2\mu C \cos\theta r^{-2}$. Thus for general shapes of particle in unbounded fluid exerting a force $\mathbf{F}$ on the fluid, at large distances

$$\psi \sim \frac{Fr}{8\pi\mu}\sin^2\theta \quad p - p_\infty \sim \frac{F}{4\pi r^2}\cos\theta \quad \mathbf{u} \sim \frac{F}{8\pi\mu r}(2\cos\theta,\, -\sin\theta,\, 0) \quad (3.14)$$

This solution for $\mathbf{u}$ and p satisfies the Stokes equations everywhere except at $r = 0$ and corresponds to a point force $\mathbf{F}$ acting at $r = 0$. It is called a **Stokeslet velocity field**.

Sedimentation and the rise of bubbles

A spherical particle of radius a and density ρ_p feels a gravitational force $\frac{4}{3}\pi a^3(\rho_p - \rho)\mathbf{g}$. This must balance the Stokes drag, so that we deduce that its sedimentation velocity is

$$\mathbf{u} = \frac{2}{9}\frac{a^2}{\mu}(\rho_p - \rho)\,\mathbf{g}. \quad (3.15)$$

If instead of a solid sphere we have a gas bubble, the solution is slightly different. We must assume that surface tension is strong enough to keep the shape spherical, but the appropriate boundary condition is now $e_{r\theta} = 0$ rather than $u_\theta = 0$. This leads to a solution

$$\psi = \tfrac{1}{2}U_0 ar\sin^2\theta\ . \quad (3.16)$$

As the coefficient of $r\sin^2\theta$ is $\frac{2}{3}$ of the value in the solid sphere case (3.9), we deduce that the drag on the bubble is $-4\pi a\mu\mathbf{U}$. A small spherical gas bubble rises with speed $\frac{1}{3}\rho g a^2/\mu$, (although for air in water the radius must be tiny for the Reynolds number to be low.)