M3A10: §1.2 Dynamics and the stress tensor

The forces that act on a fluid element are of two kinds:

(i) Volume or body forces, having long range, that are proportional to the *volume* of a fluid element (e.g. gravity).

(ii) Surface tractions, having short range, that are proportional to the *surface area* of a fluid element. Let $\mathbf{n}\,dS$ be an arbitrary element of area drawn in the fluid at $(\mathbf{x}, t)$. We write the force exerted *by* the fluid on the +side of dS *on* the fluid on the −side as $\boldsymbol{\tau} dS$. $\boldsymbol{\tau}$ is called the *surface traction* and depends not only on $\mathbf{x}$ and t, but also upon $\mathbf{n}$. We define the **Cauchy stress tensor** $\sigma_{ij}(\mathbf{x}, t)$ such that σ_{i1} is the force acting on a small surface element whose normal points in the 1-direction, and similarly for σ_{i2} and σ_{i3}.

Theorem 1: Tractions and Stress

We claim that for a general surface element with normal $\mathbf{n}$,

$$\tau_i = \sigma_{ij} n_j \ .$$

Proof. Let $V(t)$ be an arbitrary material volume of fluid having surface $S(t)$. The momentum of the fluid in $V(t)$ is then

$$\int_{V(t)} \rho\, \mathbf{u}\, dV.$$

Thus the equation of motion for the fluid in $V(t)$ is

$$\frac{d}{dt}\int_{V(t)} \rho\, \mathbf{u}\, dV = \int_{V(t)} \mathbf{F}\, dV + \int_{S(t)} \boldsymbol{\tau}\, dS \ . \tag{1.12}$$

Now suppose that V is small, having linear dimension ε. Then because volume integrals have size $O(\varepsilon^3)$, whereas surface integrals are $O(\varepsilon^2)$, then in the limit $\varepsilon \to 0$ the surface terms in equation (1.12) must balance at leading order, and so

$$\lim_{\varepsilon \to 0} \int_{S(t)} \boldsymbol{\tau}\ dS = 0.$$

Now take V to be instantaneously a small tetrahedron as sketched, with a sloping face having area dS and normal $\mathbf{n}$. The other faces have areas

$$n_1 dS, \quad n_2 dS, \quad n_3 dS$$

Because the surface forces on this tetrahedron must balance,

$$\tau_i dS + \sigma_{i1}(-n_1)dS + \sigma_{i2}(-n_2)dS + \sigma_{i3}(-n_3)dS = 0.$$

Hence, as required,

$$\tau_i = \sigma_{ij} n_j \ . \tag{1.13}$$

Theorem 2: Symmetry of the stress tensor

Provided no 'body couples' act on the fluid,

$$\sigma_{ij} = \sigma_{ji}. \tag{1.14}$$

Proof. Taking the origin to lie instantaneously within $V(t)$, the angular momentum of fluid in V is

$$\int_{V(t)} \rho \mathbf{x} \wedge \mathbf{u} \, dV,$$

and this has magnitude $O(\varepsilon^4)$. Now conservation of angular momentum implies that (in the absence of any 'body couples' e.g. a ferrofluid in a magnetic field)

$$\frac{d}{dt}\int_{V(t)} \rho \mathbf{x} \wedge \mathbf{u} \, dV = \int_{V(t)} \mathbf{x} \wedge \mathbf{F} \, dV + \int_{S(t)} \mathbf{x} \wedge \boldsymbol{\tau} \ dS \ . \tag{1.15}$$

The last term here has magnitude $O(\varepsilon^3)$, and is therefore larger than the other two terms. Thus at leading order it must vanish, i.e.

$$\lim_{\varepsilon \to 0} \int_{S(t)} \mathbf{x} \wedge \boldsymbol{\tau} dS = 0.$$

Using theorem 1, the ith component of this equation may be written

$$\int_S \varepsilon_{ijk} x_j \sigma_{km} n_m \, dS = \int_V \varepsilon_{ijk} \frac{\partial}{\partial x_m}(x_j \sigma_{km}) \, dV$$
$$= \int_V \varepsilon_{ijk} x_j \frac{\partial \sigma_{km}}{\partial x_m} \, dV + \int_V \varepsilon_{ijk} \sigma_{kj} \, dV.$$

The integrals here have magnitude $O(\varepsilon^4)$, and $O(\varepsilon^3)$ respectively, so, in the limit $\varepsilon \to 0$,

$$\varepsilon_{ijk}\sigma_{kj} = 0 \quad \Longrightarrow \quad \varepsilon_{ilm}\varepsilon_{ijk}\sigma_{kj} = 0 \quad \Longrightarrow \quad \sigma_{ml} - \sigma_{lm} = 0$$

and thus the stress tensor is symmetric. Alternatively, we can balance the moments of the surface tractions on a cube of side ε, as in lectures.

Equation of motion for a Newtonian fluid

Assuming the fluid is **homogeneous and at rest**, it can have no preferred direction. Now a general stress tensor, σ_{ij} will in general have three eigenvectors, or special directions. It follows that when the fluid is at rest, the stress tensor can be written

$$\sigma_{ij} = -p\delta_{ij} \qquad \text{where we shall call} \quad p(\mathbf{x}, t) \qquad \text{the pressure.}$$

When motion does occur, we shall write

$$\sigma_{ij} = -p\delta_{ij} + \sigma'_{ij} \qquad \text{where} \quad p = -\tfrac{1}{3}\sigma_{ii} \ . \tag{1.16}$$

σ'_{ij} is called the **deviatoric stress**. In the **inviscid fluid mechanics** course M3A2, it is assumed that $\sigma'_{ij} \equiv 0$. More generally, we will assume a linear relationship. between the deviatoric stress σ'_{ij} and the velocity gradient tensor $\partial u_i/\partial x_j$. This assumption, which is like Hooke's Law for springs, is what defines a **Newtonian fluid**. The linearity relation between the two second-order tensors takes the form

$$\sigma'_{ij} = A_{ijkl}\frac{\partial u_k}{\partial x_l} \ , \tag{1.17}$$

where A_{ijkl} is a 4th order tensor, which at first glance contains 81 unknown constant coefficients! However, remember that the fluid is **isotropic** and has no preferred direction. This means that the tensor A_{ijkl} must be **invariant** with respect to rotations of the coordinate axes. It turns out (see, for example, 'Cartesian Tensors' by Jeffreys) that this requires

$$A_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \mu'\delta_{il}\delta_{jk}$$

and so

$$\sigma'_{ij} = \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\frac{\partial u_i}{\partial x_j} + \mu'\frac{\partial u_j}{\partial x_i} \ .$$

where λ, μ and μ' are unknown constants. Furthermore, two of these can be eliminated. Firstly, as we are dealing with incompressible fluids, $\nabla \cdot \mathbf{u} = 0$ and λ is irrelevant. Secondly, we showed above that $\sigma_{ij} = \sigma_{ji}$. This means that $\mu' = \mu$. We deduce that the stress tensor σ_{ij} for a Newtonian fluid is

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij} \qquad \text{where} \quad e_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \ . \tag{1.18}$$

The constant μ is known as the **viscosity** of the fluid.

The momentum equation for a viscous fluid

With the expression (1.13) for the surface traction $\boldsymbol{\tau}$, the equation of motion (1.12) for $V(t)$ becomes

$$\frac{d}{dt}\int_{V(t)} \rho\, u_i\, dV = \int_{V(t)} F_i\, dV + \int_{S(t)} \sigma_{ij} n_j\, dS.$$

Using Reynolds' transport theorem (1.3), mass conservation (1.4) and the divergence theorem (0.8) then gives

$$\int_{V(t)} \rho\, \frac{Du_i}{Dt}\, dV = \int_{V(t)} F_i\, dV + \int_{V(t)} \frac{\partial \sigma_{ij}}{\partial x_j}\, dV.$$

and because V is arbitrary,

$$\rho\, \frac{Du_i}{Dt} = F_i + \frac{\partial \sigma_{ij}}{\partial x_j} \; . \qquad (1.19)$$

This is the Cauchy momentum equation for the fluid, and holds for any stress tensor. Now for the Newtonian stress relation (1.18), we have

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \mu \frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i$$

and so, from (1.19), the momentum equation for a Newtonian Fluid is

$$\rho\, \frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{F} + \mu \nabla^2 \mathbf{u} \; .$$

We can now, finally, write down the **Navier-Stokes equations,** which we will solve for the remainder of this course. For constant density ρ and viscosity μ, and external force $\mathbf{F}$, we have

The Navier-Stokes Equations for an incompressible fluid

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla p + \mathbf{F} + \mu \nabla^2 \mathbf{u} \; . \qquad (1.20)$$

$$\nabla \cdot \mathbf{u} = 0 \; . \qquad (1.21)$$

When $\mu = 0$ we obtain the **Euler equations**, which are the basis of M3A2. Note, however, that as $\mu \to 0$, the fluid does not necessarily behave in the same way as it would if $\mu = 0$.