M3A10 Viscous Flow: Some useful vector identities

This sheet can be found on the Web: http://www.ma.ic.ac.uk/~ajm8/M3A10

Most of this sheet is revision from M2M1. We begin by recalling the suffix notation for vectors and tensors, and then present some vector identities, which we write both in vector notation and suffix (tensor) notation. In different circumstances each is useful, and you should try to be at ease switching from one to another.

A vector, $\mathbf{u}$, has by default three components u_1, u_2 and u_3. We write u_i as an alternative representation of $\mathbf{u}$, where the index i is allowed to take the values 1, 2 and 3 in turn. We represent three-dimensional space by the Cartesian coordinates (x_1, x_2, x_3) so that $\mathbf{x}$ or equivalently x_i is a general position vector. Thus $\nabla\phi$ can be written $\partial\phi/\partial x_i$ in suffix notation.

The summation convention: in any product of quantities any index can appear once or twice. If only once, it takes all three values in turn. If it appears twice, then it is treated as a **dummy index to be summed over**. Thus $a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 = \mathbf{a}\cdot\mathbf{b}$, $\partial u_i/\partial x_i = \nabla\cdot\mathbf{u}$ and $a_i b_j a_i$ is equivalent to $|\mathbf{a}|^2\mathbf{b}$. You must have this idea clear in your mind. Never repeat an index three times in a single product. **In any equation each product must have exactly the same 'free' indices.**

Two special tensors: Recall the Kronecker delta δ_{ij} and the alternating tensor ε_{ijk}.

$$\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \qquad \varepsilon_{ijk} = \begin{cases} +1 & \text{if} \quad ijk = 123,\ 231,\ 312 \\ -1 & \text{if} \quad ijk = 213,\ 132,\ 321 \\ 0 & \text{otherwise} \end{cases} \tag{0.1}$$

We can then see by direct calculation that, for example,

$$(\nabla\wedge\mathbf{u})_i = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} \qquad \delta_{ij}u_j = u_i \qquad \varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\ . \tag{0.2}$$

Some vector identities:

$$\nabla\cdot(\phi\mathbf{F}) = \phi\nabla\cdot\mathbf{F} + \nabla\phi\cdot\mathbf{F} \qquad \frac{\partial}{\partial x_i}(\phi F_i) = \phi\frac{\partial F_i}{\partial x_i} + \frac{\partial\phi}{\partial x_i}F_i \tag{0.3}$$

If we write

$$\boldsymbol{\omega} = \nabla\wedge\mathbf{u} \qquad \omega_i = \varepsilon_{imn}\frac{\partial u_n}{\partial x_m}, \tag{0.4}$$

then

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\left(\tfrac{1}{2}|\mathbf{u}|^2\right) + \boldsymbol{\omega}\wedge\mathbf{u} \qquad u_j\frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\tfrac{1}{2}u_j u_j\right) + \varepsilon_{ijk}\omega_j u_k\,, \tag{0.5}$$

$$\nabla^2\mathbf{u} = \nabla(\nabla\cdot\mathbf{u}) - \nabla\wedge\boldsymbol{\omega} \qquad \frac{\partial^2 u_i}{\partial x_j\partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) - \varepsilon_{ijk}\frac{\partial\omega_j}{\partial x_k} \tag{0.6}$$

and

$$\mathbf{u}\cdot(\nabla\wedge\boldsymbol{\omega}) = \nabla\cdot(\boldsymbol{\omega}\wedge\mathbf{u}) - |\boldsymbol{\omega}|^2 \qquad u_i\varepsilon_{ijk}\frac{\partial\omega_k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\varepsilon_{ijk}\omega_k u_i\right) - \omega_k\omega_k\ . \tag{0.7}$$

The divergence theorem: this is perhaps the most important result in all mathematics. We shall use it in the following form. If V is a volume enclosed by a surface S with outward unit normal $\mathbf{n}$, and T is **any** tensor (i.e. T could be a scalar, a vector, a second order tensor…), then

$$\boxed{\int_V \frac{\partial T}{\partial x_j}\, dV = \int_S n_j T\, dS} \tag{0.8}$$

If we replace T by the vector u_j, for example, we obtain the ordinary divergence theorem,

$$\int_V \nabla \cdot \mathbf{u}\, dV = \int_S \mathbf{n} \cdot \mathbf{u}\, dS$$

If we replace T by the scalar p, we find

$$\int_V \nabla p\, dV = \int_S p\, \mathbf{n}\, dS \tag{0.9}$$

A few facts about tensors: We say a second order tensor σ_{ij} is symmetric if $\sigma_{ij} = \sigma_{ji}$, or antisymmetric if $\sigma_{ij} = -\sigma_{ji}$.

Any second order tensor a_{ij} can be written as the sum of a symmetric tensor e_{ij} and an antisymmetric tensor Ω_{ij}

$$a_{ij} = e_{ij} + \Omega_{ij} \qquad \text{where} \quad e_{ij} = \tfrac{1}{2}(a_{ij} + a_{ji}), \qquad \Omega_{ij} = \tfrac{1}{2}(a_{ij} - a_{ji}) \tag{0.10}$$

A **symmetric** tensor has real eigenvalues and its eigenvectors are mutually orthogonal. By a change of basis (rotation of the coordinate axes) it can be **diagonalised**, i.e. in a suitable coordinate system

$$e_{ij} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}_{ij} \tag{0.11}$$

The axes are then said to point along the **principal axes** of the tensor e_{ij}. These axes always have physical significance.

An antisymmetric tensor has only 3 independent components, and can be related to a vector: (the factor of -2 is included for later convenience)

$$-2\Omega_{ij} = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}_{ij} = \varepsilon_{ijk}\omega_k \ . \tag{0.12}$$