M^3_4A21 Handout 5: Motion through Fluids: Lift and Drag.

In order to understand the mechanics of swimming and flying, it is necessary to understand the forces which act on a body when it moves through a fluid, typically air or water. Fluid dynamics is a rich and complex subject. On this sheet we attempt a very brief summary of the bulk force acting on the body due to the fluid.

All points in the fluid have their own velocity, so that we represent the velocity by the vector field, $\mathbf{v}(\mathbf{x}, t)$. Also, fluids have a **pressure** distribution $p(\mathbf{x}, t)$. Variations in pressure give rise to a force, $-\nabla p$, which points from regions of high pressure to those of low pressure. The fluid is then characterised by two quantities, which we will assume to be constant, namely its **density**, ρ, and its **viscosity**, μ. It is also common to refer to the **kinematic viscosity** $\nu \equiv \mu/\rho$. A **Newtonian** fluid then obeys the momentum equation

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p + \mathbf{F} + \mu\nabla^2\mathbf{v} \tag{7.1}$$

while mass conservation for an **incompressible** fluid requires

$$\nabla\cdot\mathbf{v} = 0\ . \tag{7.2}$$

In (7.1) $\mathbf{F}$ is an external body force, for example gravity gives rise to $\mathbf{F} = -\rho g\widehat{\mathbf{z}}$, where $\widehat{\mathbf{z}}$ points vertically upwards. (7.1) and (7.2) are called the **Navier-Stokes** equations. They are **non-linear** because of the $\mathbf{v}\cdot\nabla\mathbf{v}$ part of the acceleration, or **inertial** term on the left of (7.1).

When a body moves relative to the surrounding fluid it must do work for two reasons. Firstly, it must move the fluid out of its way, thus imparting kinetic energy to the fluid. Secondly, it must overcome the internal friction of the fluid, as measured by the **viscous** term in (7.1). The relative importance of these two effects can be estimated by looking at a typical ratio of the inertial to viscous forces,

$$\frac{|\rho\mathbf{v}\cdot\nabla\mathbf{v}|}{|\mu\nabla^2\mathbf{v}|} \simeq \frac{\rho V_0^2/L}{\mu V_0/L^2} = \frac{\rho V_0 L}{\mu} = R\ . \tag{7.3}$$

Where we have estimated the size of the terms rather crudely, letting V_0 be a typical magnitude of $|\mathbf{v}|$ and L be a typical length-scale of variation. This ratio R is known as the **Reynolds number** of the flow. The whole nature of the motion depends crucially on the size of this number. Flows which are small-scale, slow and sticky have **Low Reynolds Number**, whereas motions which are large, fast and momentum dominated have a **High Reynolds Number.** For swimming in water, $R \sim 10^{-5}$ for a bacterium, $R \sim 0.1$ for a protozoan, $R \sim 3\times 10^5$ for a medium sized fish and $R \sim 2\times 10^6$ for a human. The idea of **dynamics similarity** comes from this: for the fluid dynamics, only the value of R is important. A protozoan swimming in water is just like a human swimming through syrup at 0.1m/s, apart from shape differences.

Note that although we are interested in a body moving in stationary fluid, it is mathematically equivalent to consider the body at rest and the fluid moving past it.

Low Reynolds Number Flows:

When the inertial term is much smaller than the viscous term, the nonlinear term in (7.1) can be neglected. This linear problem is fairly easy to solve. A body moving with speed $\mathbf{V}_0$ through a fluid when $R \ll 1$ experiences a **drag** force

$$\mathbf{D} = 6\pi L\mu\, M\mathbf{V}_0 \qquad \text{where } M \text{ is a dimensionless geometrical tensor (matrix).} \qquad (7.4)$$

If the direction of $\mathbf{V}_0$ is along a principle axis of M, then $M\mathbf{V}_0 = \alpha\mathbf{V}_0$, and the drag acts in the opposite direction to the velocity. For a sphere of radius L, it is found that $\alpha = 1$. α does not vary too much for other shapes. Neither does it matter very much in which orientation the body lies; the drag force tends to be governed by the greatest linear dimension of the body. This is in marked contrast to flows at higher R, where a streamlined shape is very important. (Try pushing a boat sideways!) The crucial point is that the drag force varies **linearly** with the velocity. Note also that the entire flow is reversible: if we replace $\mathbf{v}$ by $-\mathbf{v}$, then $\mathbf{D}$ goes to $-\mathbf{D}$.

High Reynolds number Flows:

The situation is much more complicated when $R \gg 1$. We would like then to ignore the viscous term totally; however, it is important very close to solid boundaries in **boundary layers.** Sometimes these thin layers **separate** leading to a marked increase in drag (when flying, this is called **stall**). It is also possible for vortices to be shed, for the force to be time-dependent and for the flow to be turbulent.

At high Reynolds numbers, the drag on a body moving with speed V_0 is approximately quadratic in the velocity, and may be summarised

$$D = \tfrac{1}{2}C_D\,\rho S V_0^2 \qquad \text{where } S \text{ is the cross-sectional area normal to } \mathbf{v}\ . \qquad (7.5)$$

C_D is a dimensionless number known as the **drag coefficient.**, which varies with both shape and Reynolds number. In this case, for the drag to be small it is very important for the shape to be **streamlined** and to present a small cross-section to the oncoming flow. (Think of a sail, where **large** drag is desirable.)

Lift forces:

At high R also it is possible for the total force not to be aligned with the velocity. (Imagine carrying a large sheet of wood slightly inclined to a strong wind.) A large force normal to the velocity can be generated, due to an asymmetric pressure distribution. This **lift** force, G, is usually

$$G = \tfrac{1}{2}C_L\,\rho V_0^2 W \qquad \text{where } W \text{ is the wing area} \qquad (7.6)$$

and C_L is the dimensionless **lift coefficient**. It is proportional to $\sin\beta$ where β is the angle between wing and oncoming flow. This lift can balance gravity and enable flight. It does however increase the drag force, by an amount known as the **induced drag**, D_I,

$$D_I = \frac{kG^2}{\frac{1}{2}\rho V_0^2 b^2} \qquad \text{where } b \text{ is the wing length and } k \simeq 0.1 \text{ depends on the shape.} \qquad (7.7)$$

The reasons for the induced drag are complex, and to do with the trailing vortices at the wing tips. The total drag acting in the presence of lift is given by the sum of (7.7) and (7.5).